Dynamics, instability, and bifurcation in the mechanics of biological growth

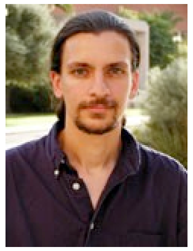
Alain Goriely

Department of mathematics, University of Arizona

In collaboration with Martine Ben Amar, Michel Destrade, Michael Tabor and


Joe McMahon


Becky Vandiver


Mark Robertson


Derek Moulton

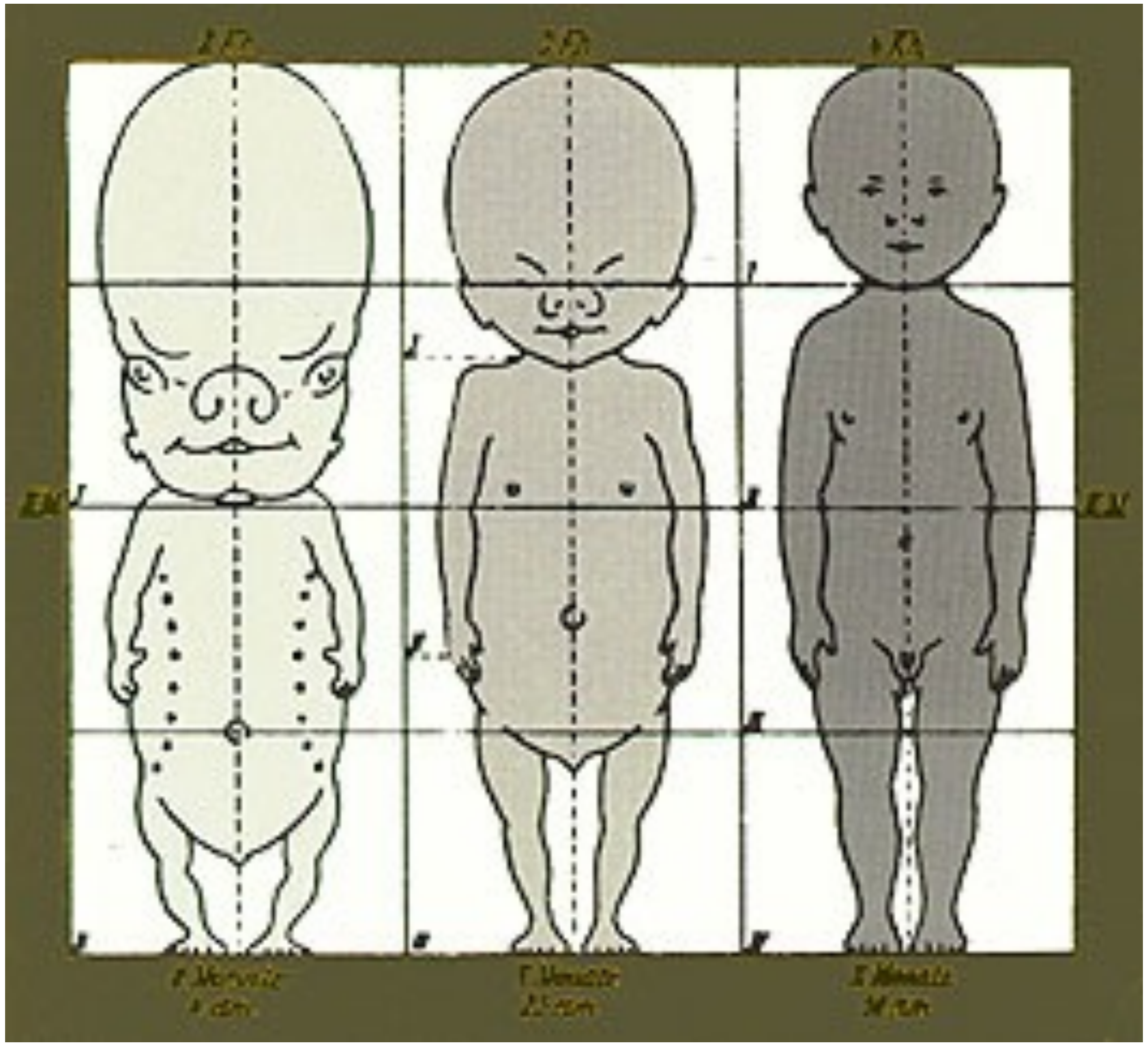
RUMMBA: Research Unit in Mechanics Mathematics Biology and its Applications

Research supported in part by the **N**ational **S**cience **F**oundation (DMS)

You will find some papers on the subject:
Goriely on **Google** (I feel lucky!).

Growth and Form

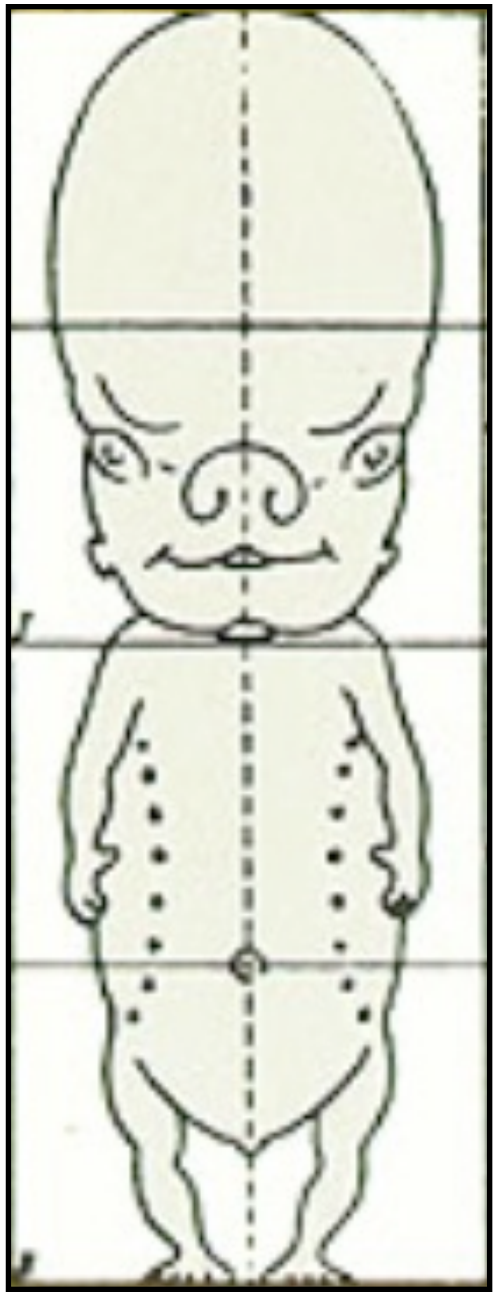
- Part 1: Introduction
- Part 2: Modeling of growth
- Part 3: Analysis of growth



Part I. Introduction

1. Basic issues

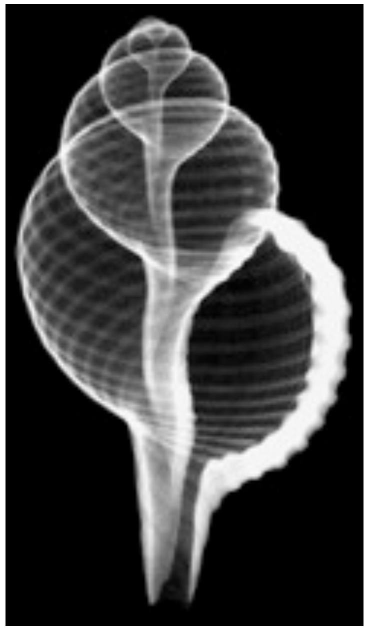
- Accretion & Volumetric growth
- Stress ⇒ Growth
- Growth ⇒ Stress
- Stress ⇒ Growth ⇒ Stress
- Mechanics & Instability

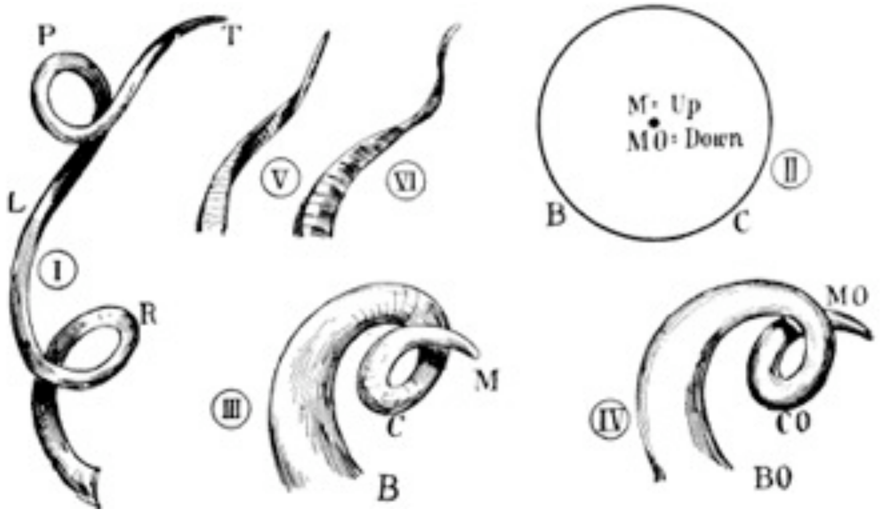


Accretion and volumetric growth

- Accretion

  Typical for hard tissues
  (seashells, horns, teeth, bones)





Ref: Cook, d'Arcy Thompson
Skalak and Hoger ('97)

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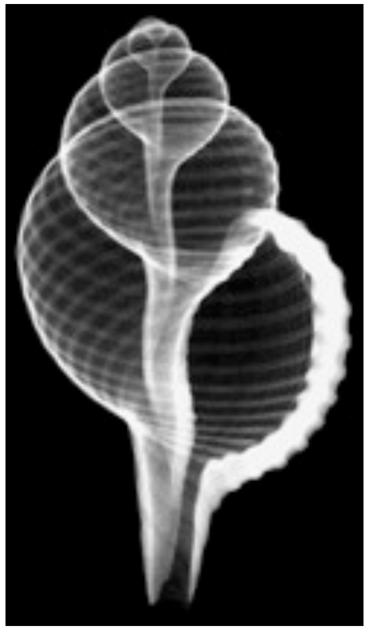

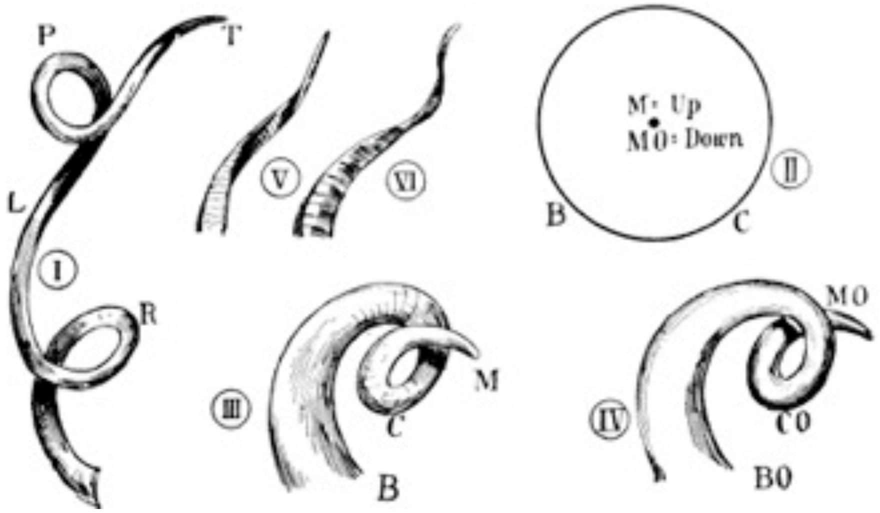

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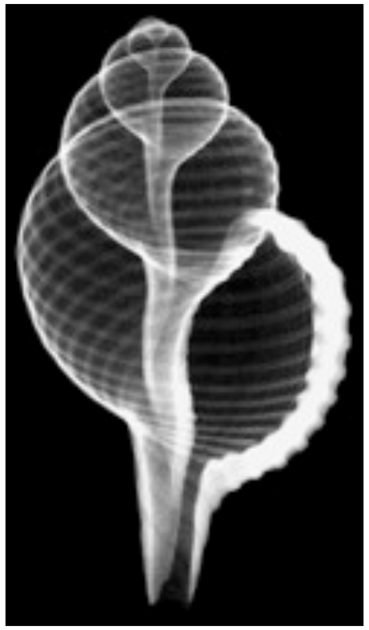

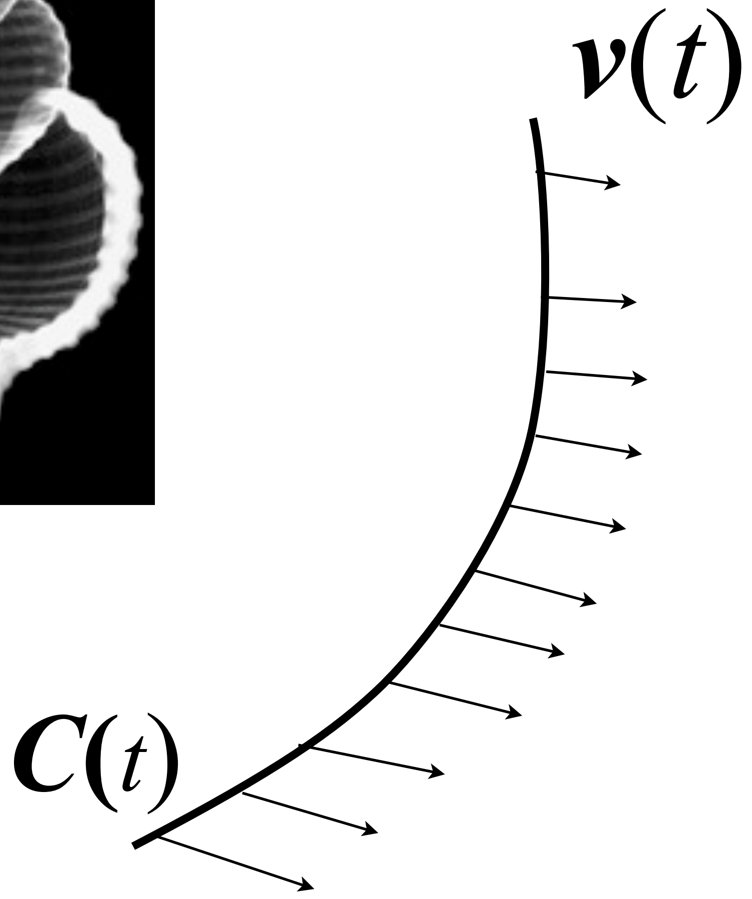

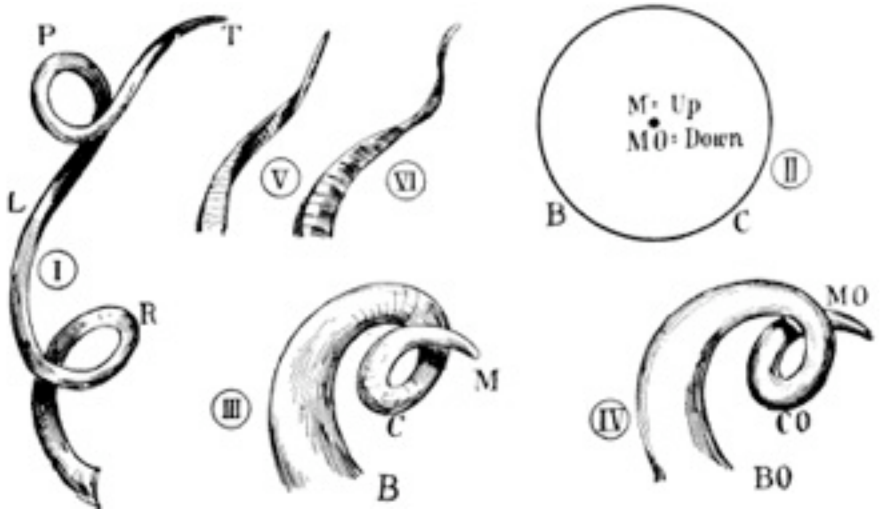

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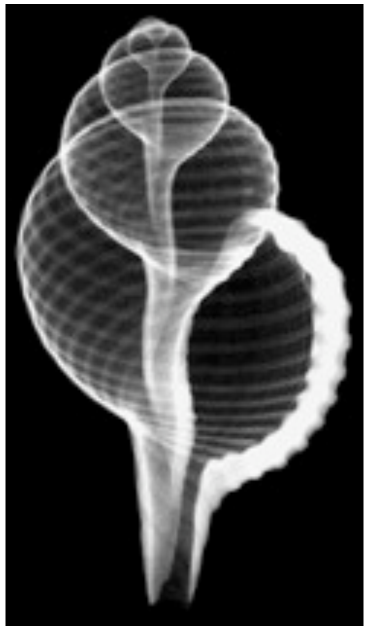

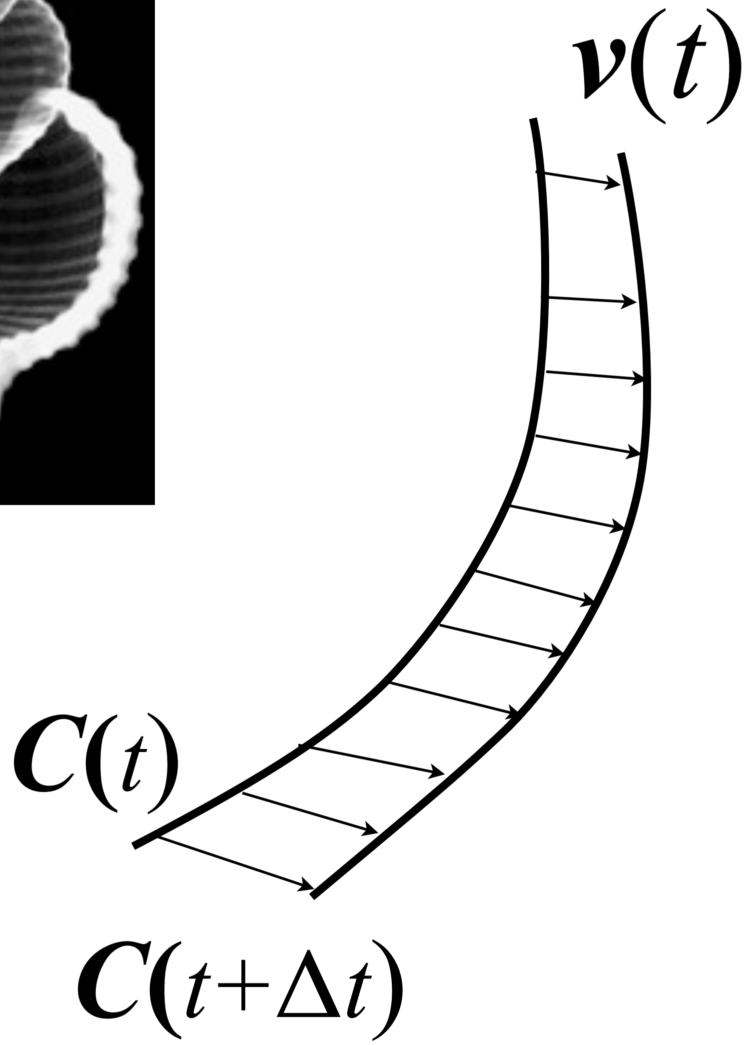

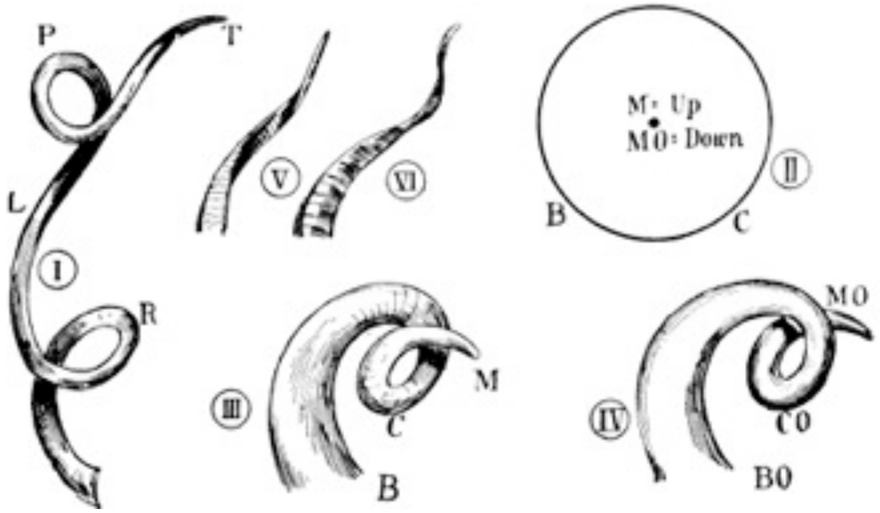

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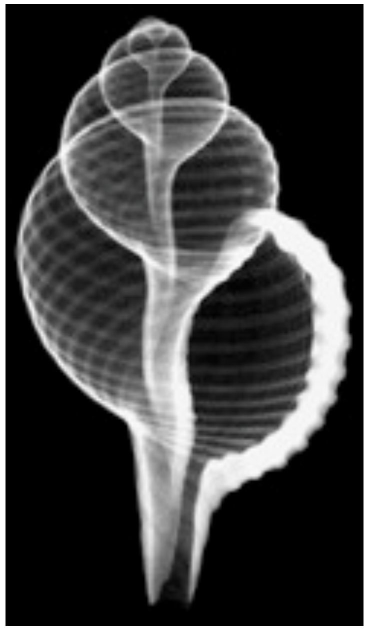

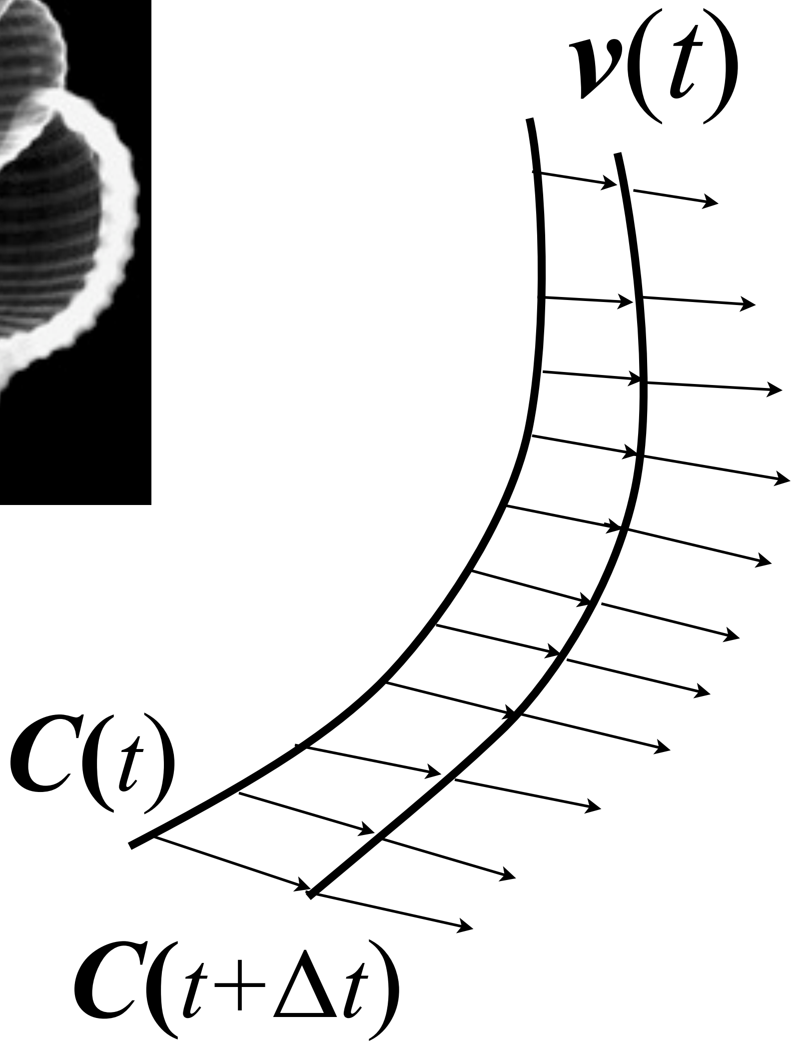

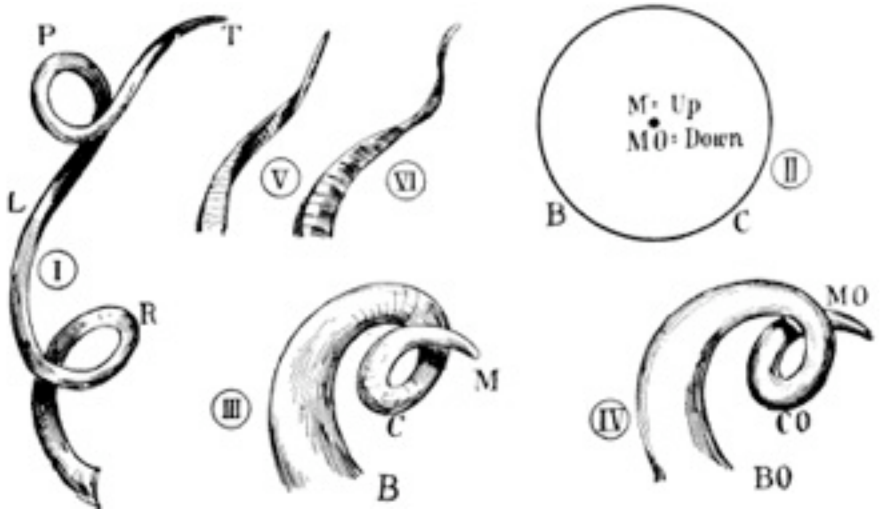

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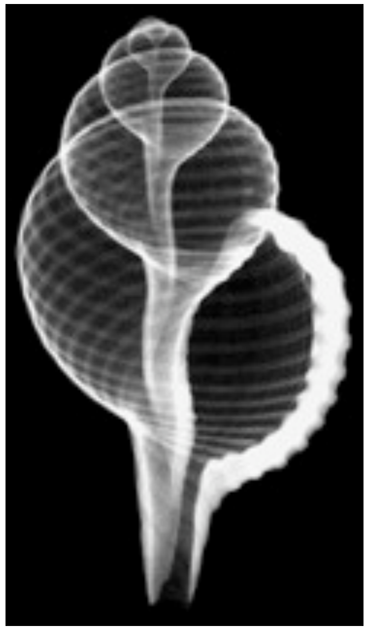

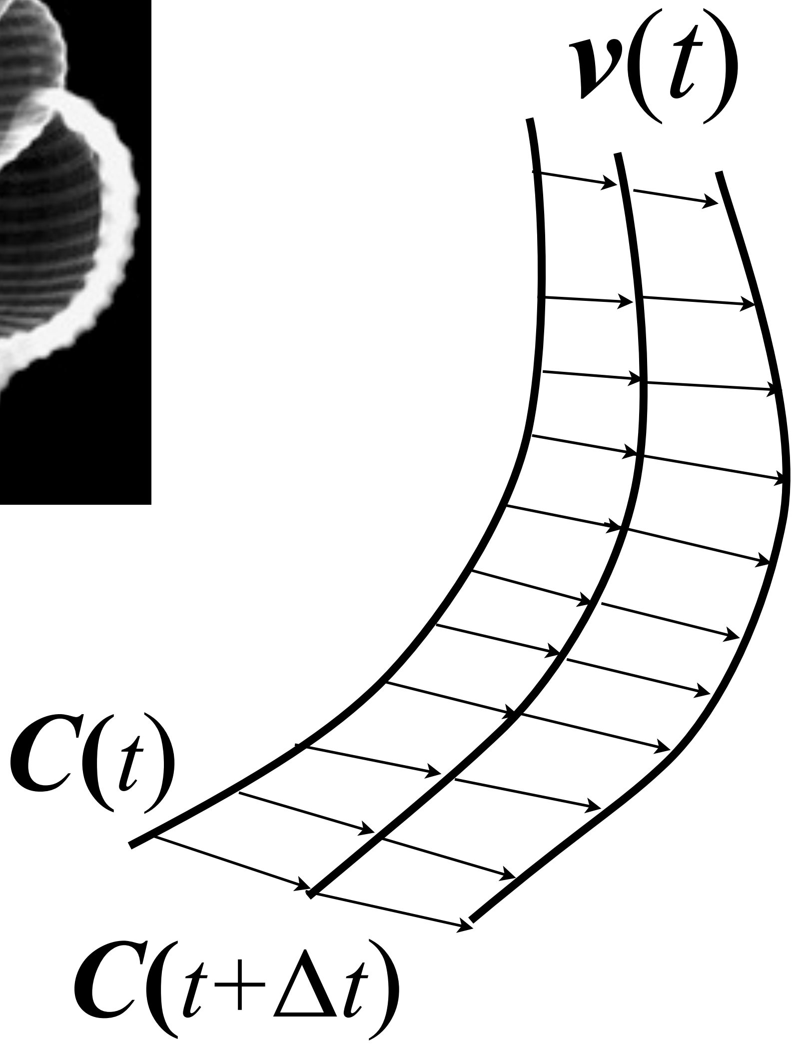

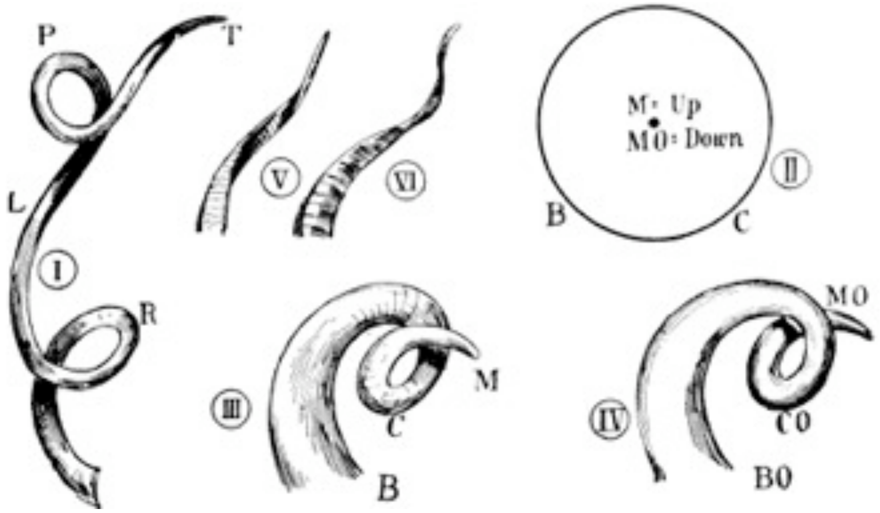

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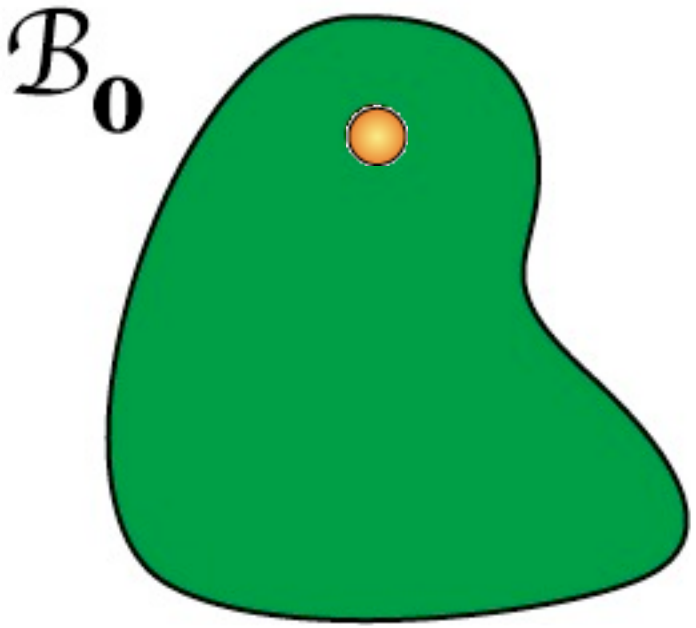

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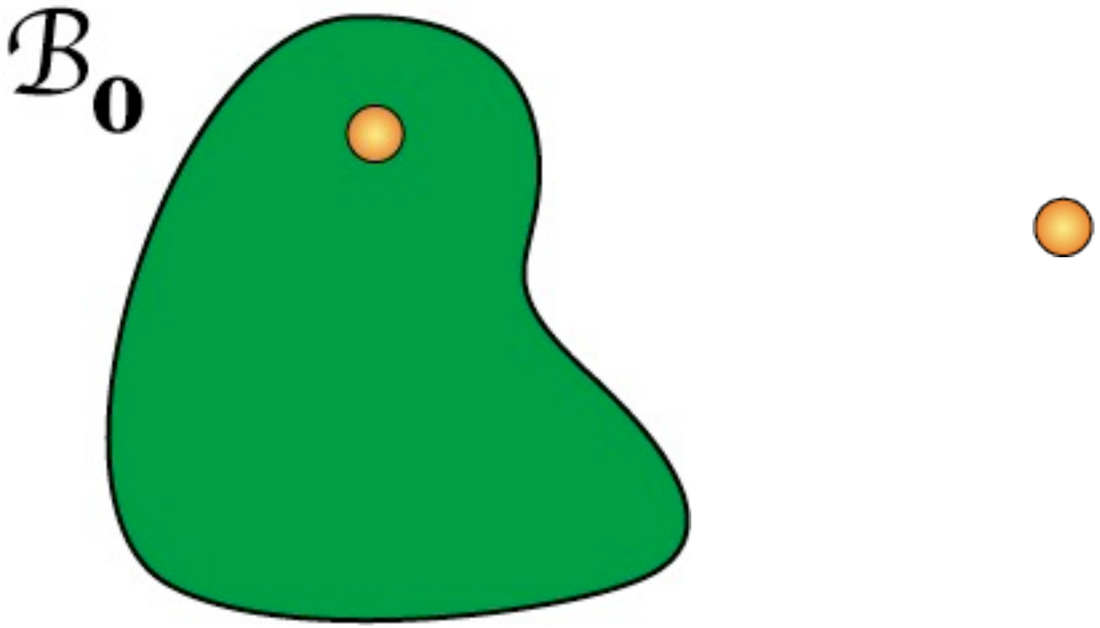

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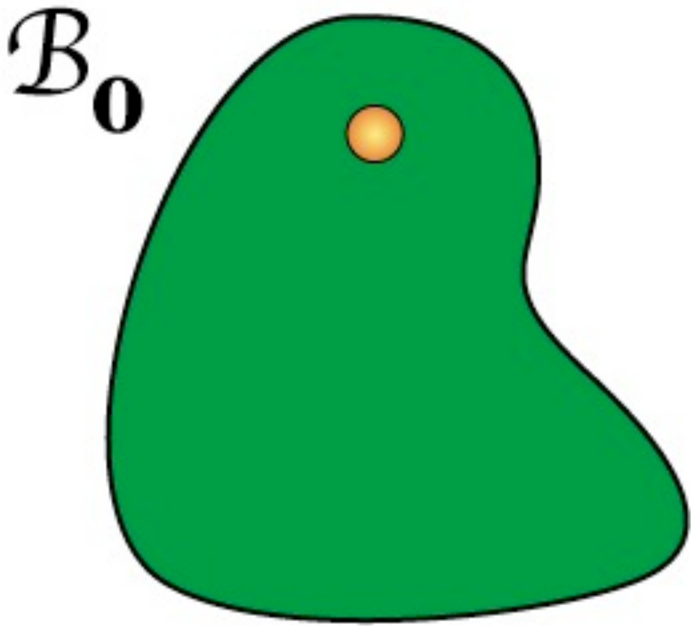
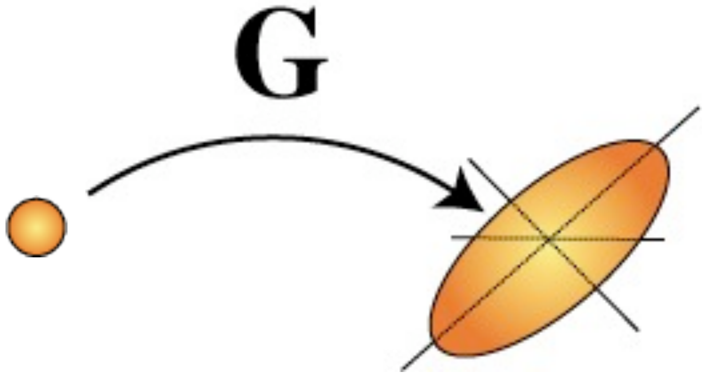


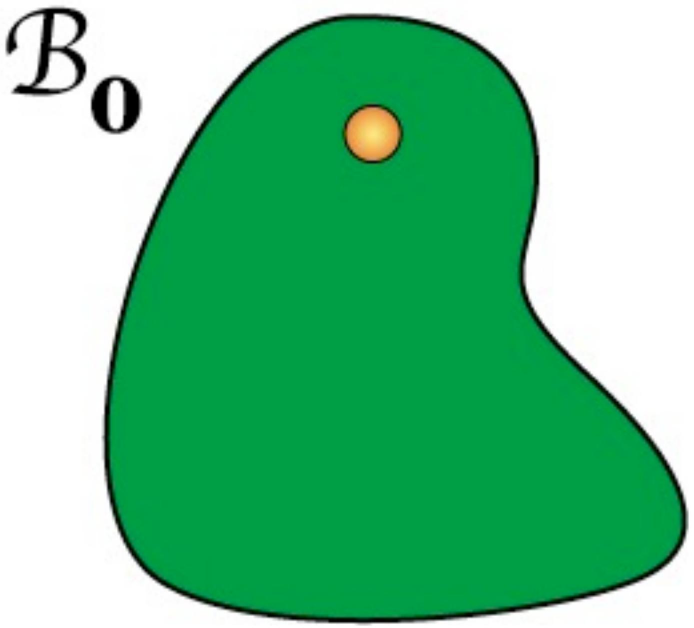


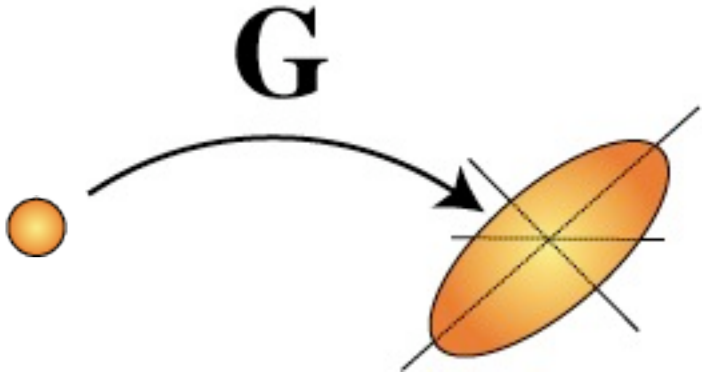
$\mathbf{G}$ describes the growth of a local volume element.

Accretion and volumetric growth

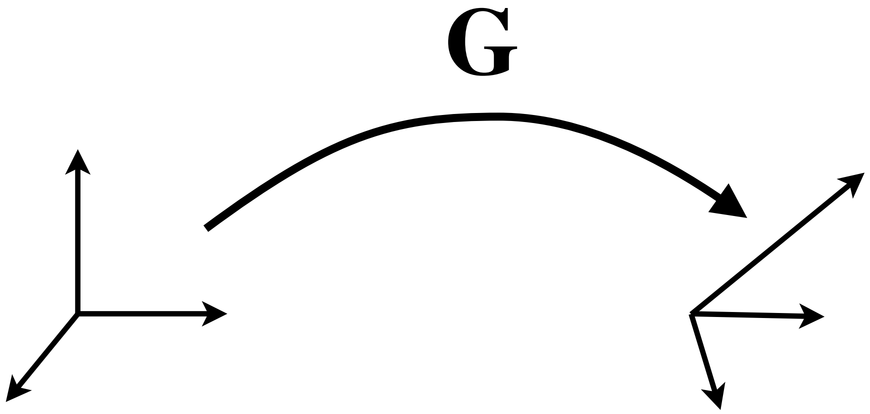
- Volumetric growth

Typical for soft tissues (heart, liver, muscles, arteries, tumours,...)



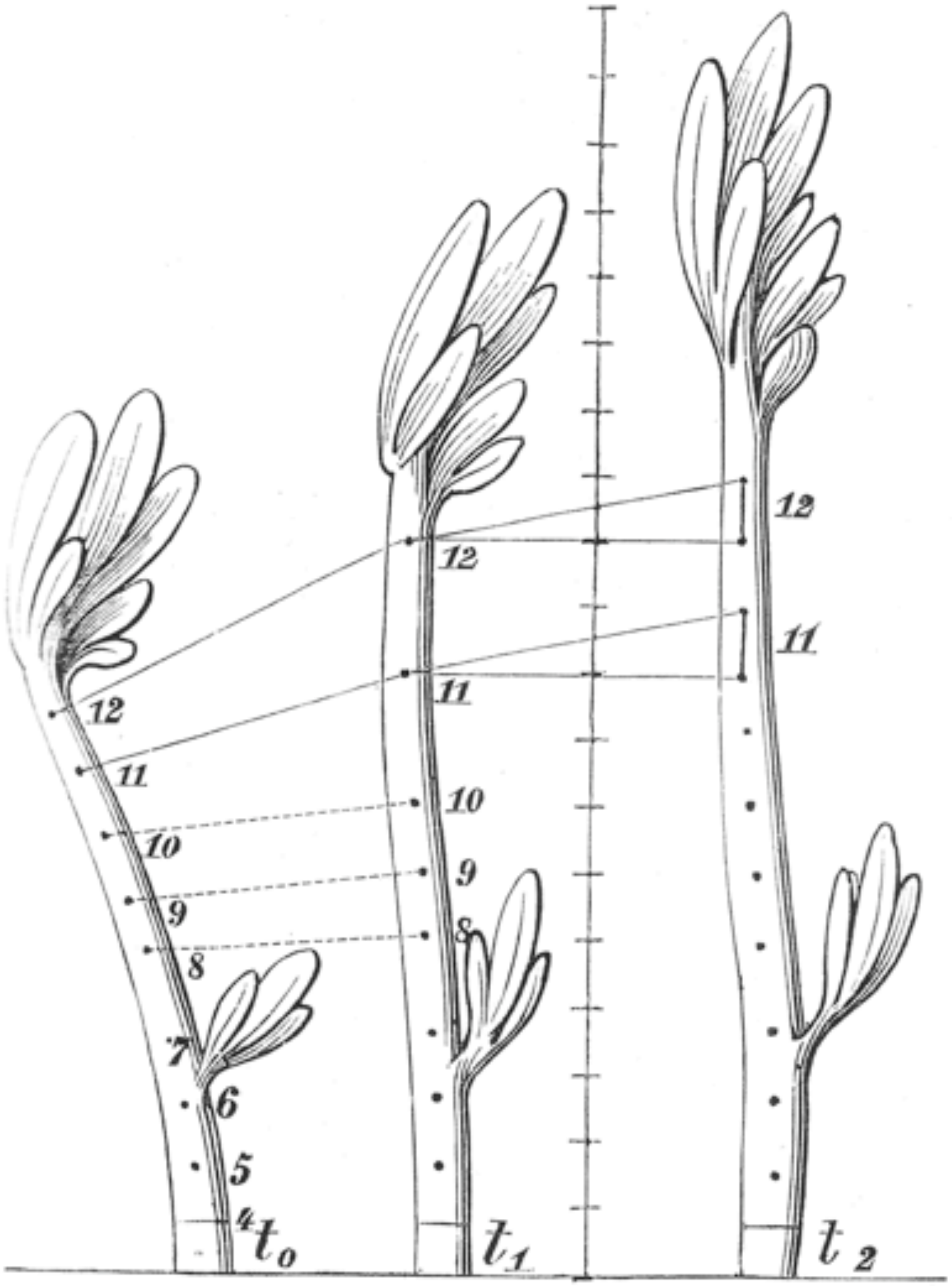


$\mathbf{G}$ describes the growth of a local volume element.



It is a linear map acting on the tangent space of our manifold.

Kinematics of growth



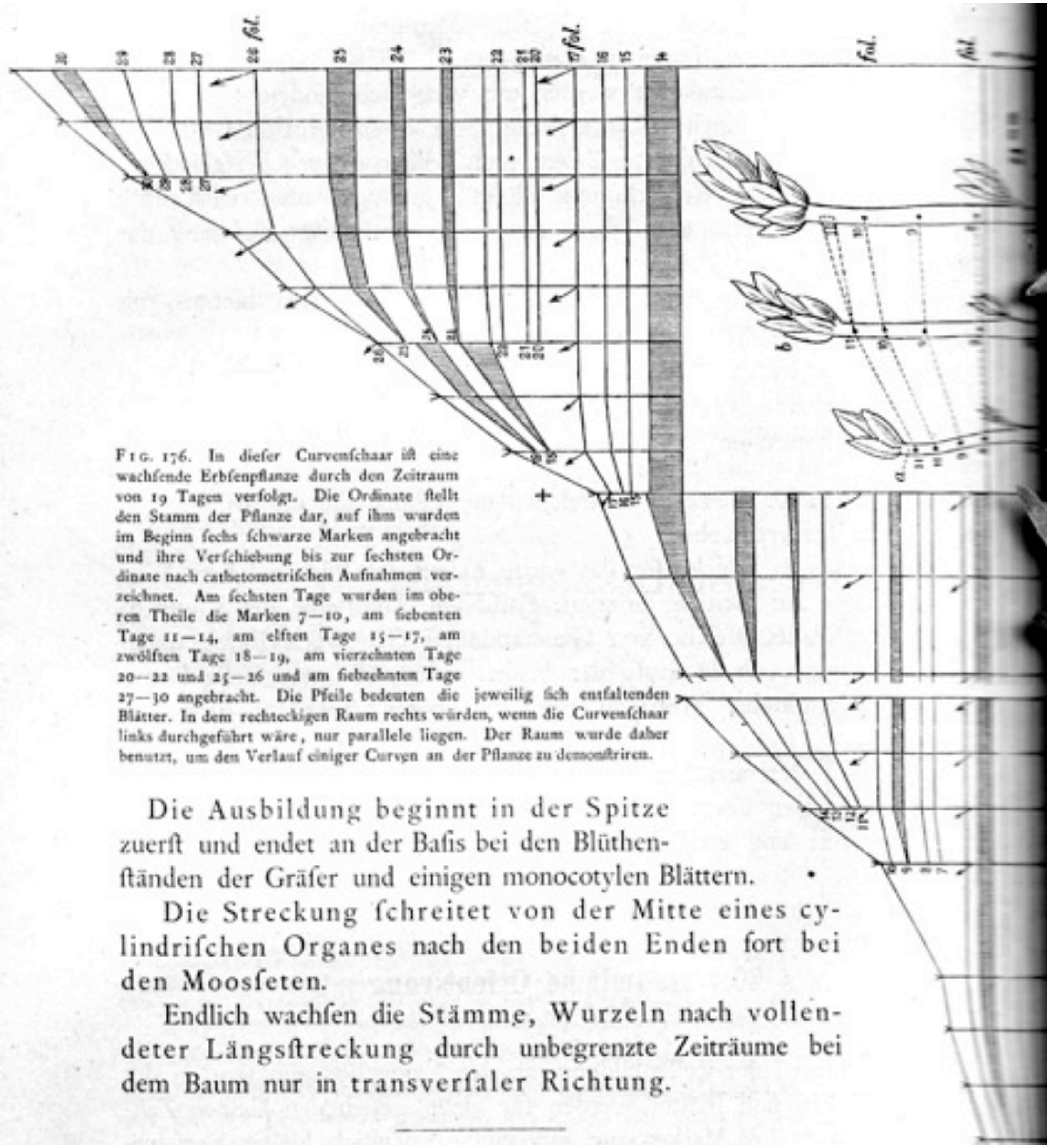


FIG. 176. In dieſer Curvenſchaar iſt eine wachſende Erbſenpflanze durch den Zeitraum von 19 Tagen verfolgt. Die Ordinate ſtellt den Stamm der Pflanze dar, auf ihm wurden im Beginn ſechs ſchwarze Marken angebracht und ihre Verſchiebung bis zur ſechsten Ordinate nach cathetometriſchen Aufnahmen verzeichnet. Am ſechsten Tage wurden im oberen Theile die Marken 7—10, am ſiebenten Tage 11—14, am elften Tage 15—17, am zwölften Tage 18—19, am vierzehnten Tage 20—22 und 25—26 und am ſiebzehnten Tage 27—30 angebracht. Die Pfeile bedeuten die jeweilig ſich entfaltenden Blätter. In dem rechteckigen Raum rechts würden, wenn die Curvenſchaar links durchgeführt wäre, nur parallele liegen. Der Raum wurde daher benutzt, um den Verlauf einiger Curven an der Pflanze zu demonſtriren.

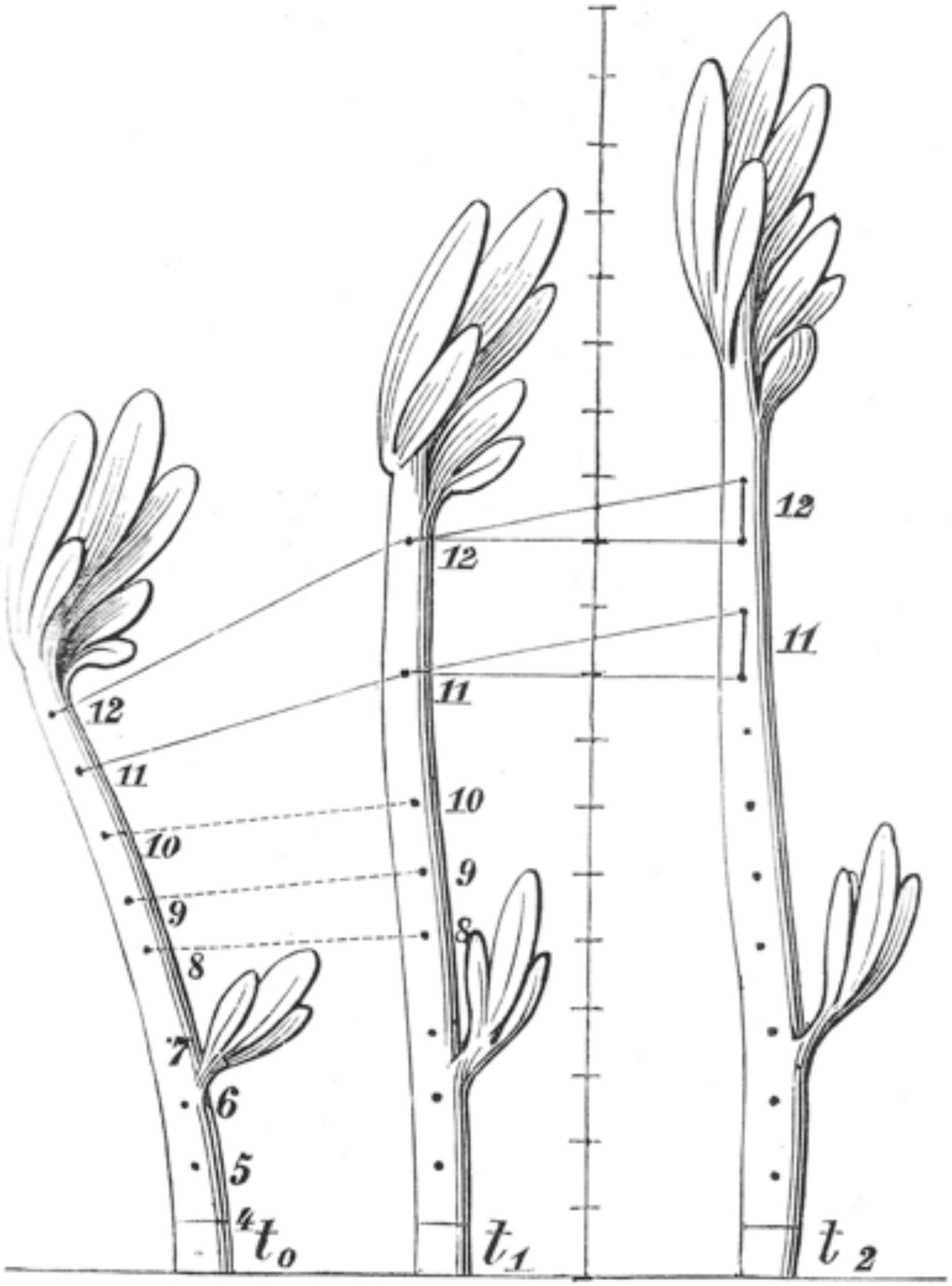
Die Ausbildung beginnt in der Spitze zuerſt und endet an der Baſis bei den Blüthenſtänden der Gräſer und einigen monocotylen Blättern.

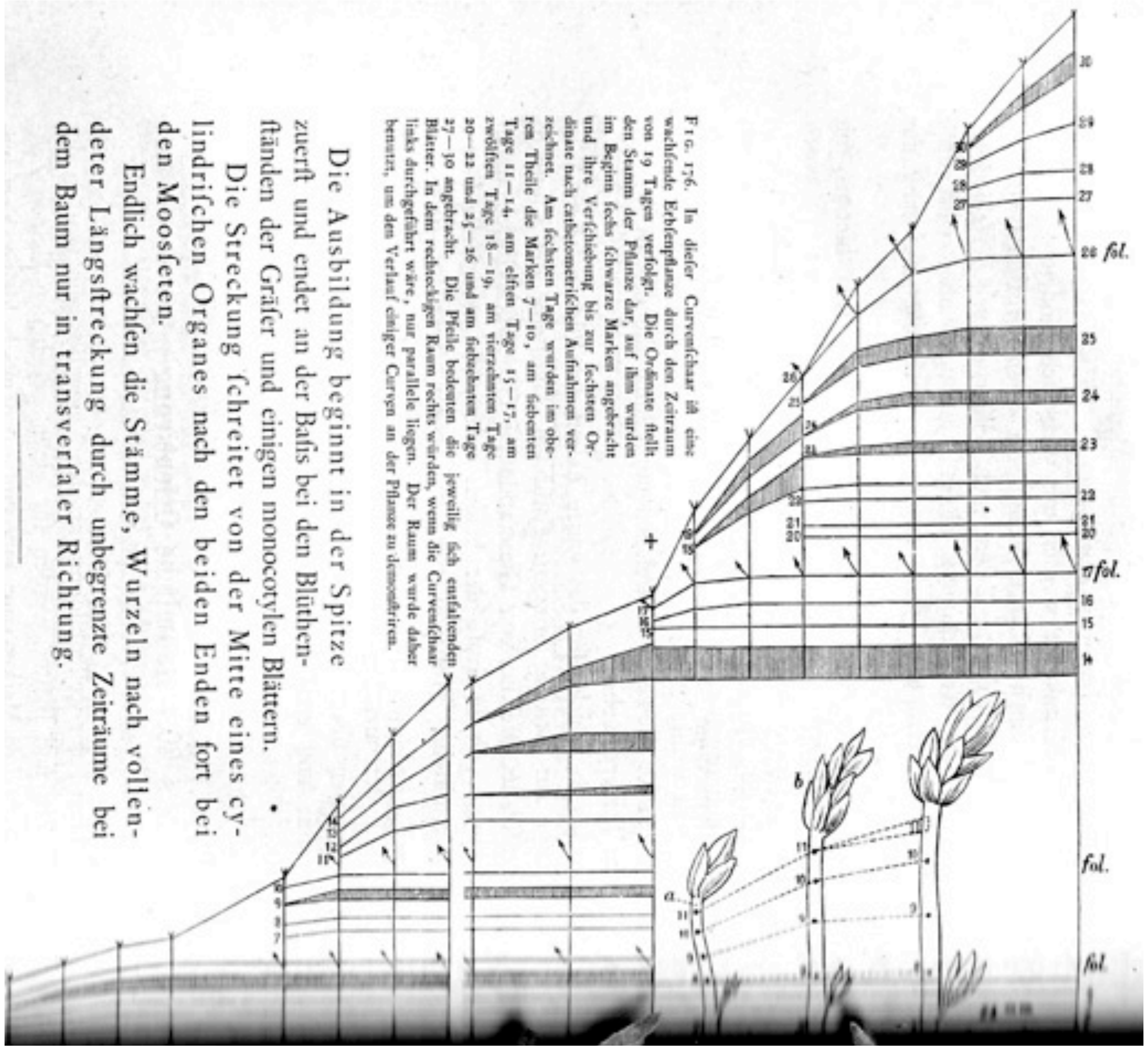
Die Streckung ſchreitet von der Mitte eines cylindriſchen Organes nach den beiden Enden fort bei den Moosſeten.

Endlich wachſen die Stämme, Wurzeln nach vollendeter Längsſtreckung durch unbegrenzte Zeiträume bei dem Baum nur in transverſaler Richtung.

Müller, *Handbuch der allgemeinen Botanik* 1880

Kinematics of growth





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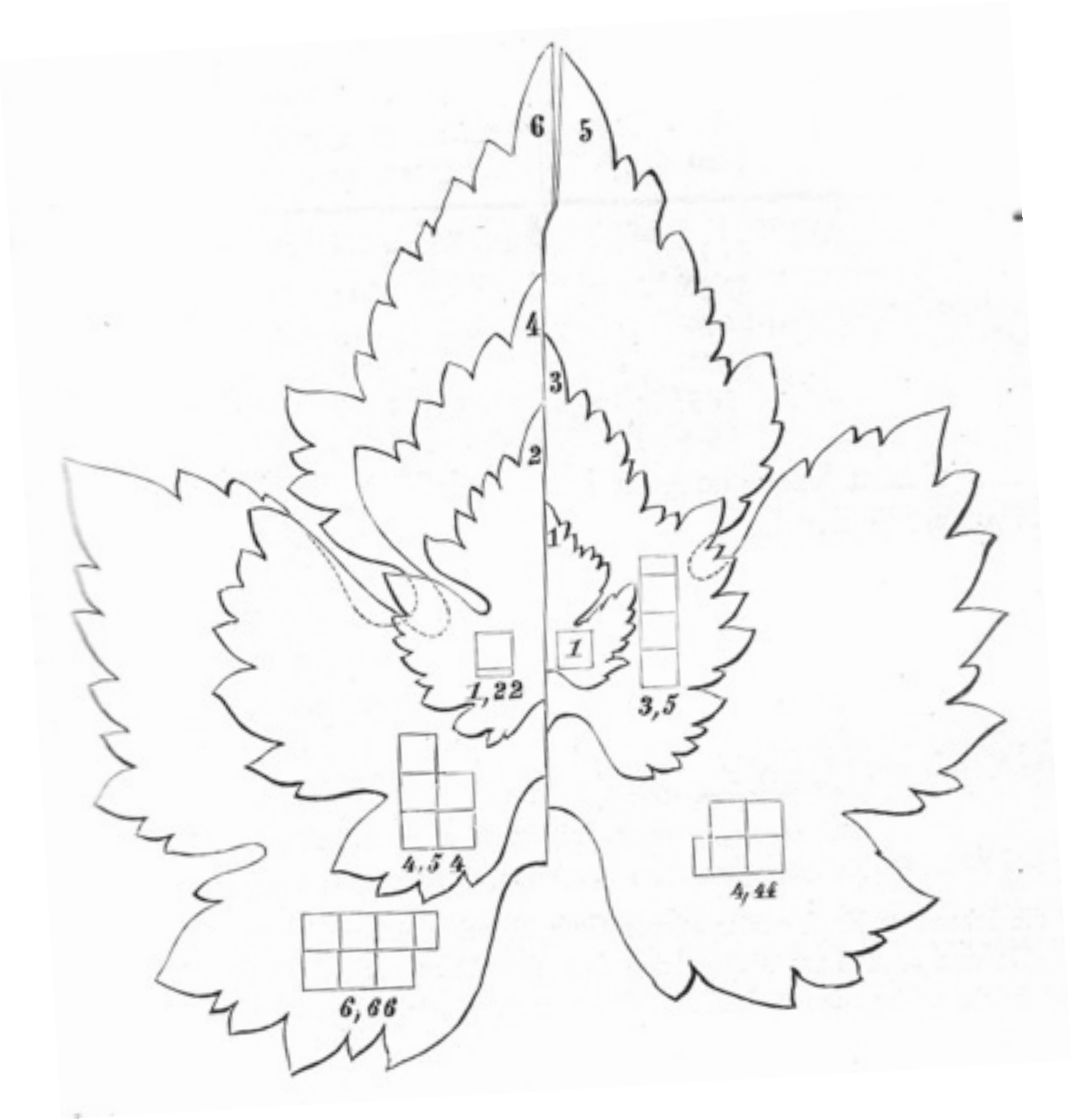
Kinematics in 2D



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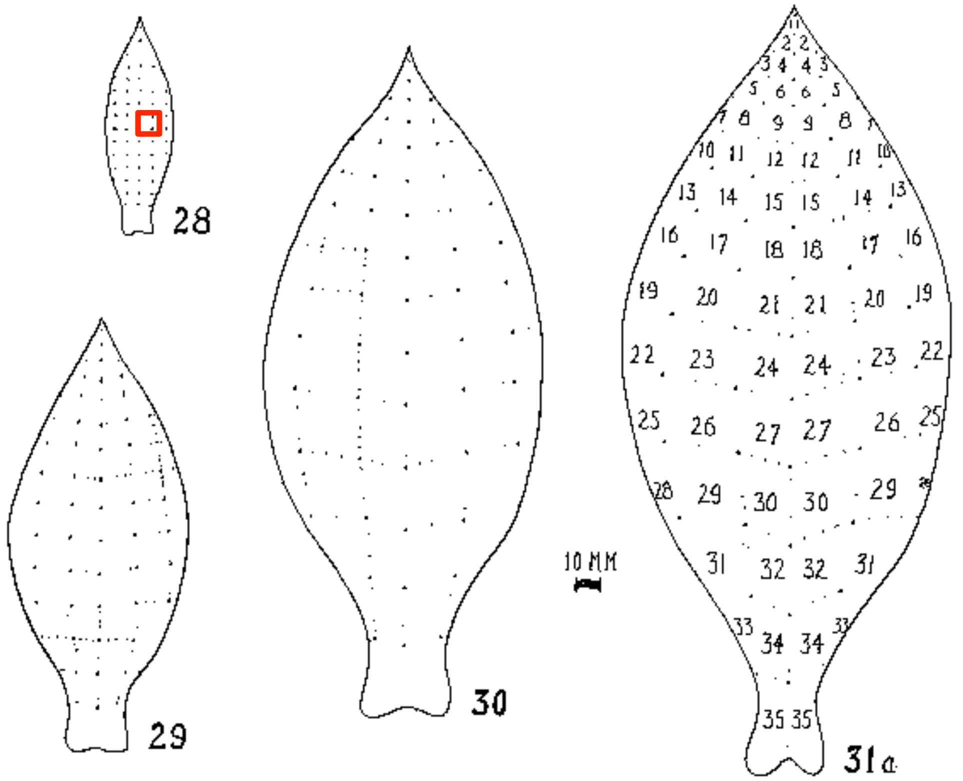

Fig. 28–31 *a*. Successive stages in the development of the leaf from the time it is ¼ its final size, to maturity. The entire leaf was marked into 5-mm. squares when 85 mm. long (fig. 28). The appearance of these segments as the leaf matures indicates the differing rates of expansion as the leaf grows to its final size. All numbered segments in fig. 31 *a*, for both right and left hand side of leaf, correspond to those in fig. 28–30. They also apply to those shown in fig. 31 *b* and 31 *c*.

Avery, 1933

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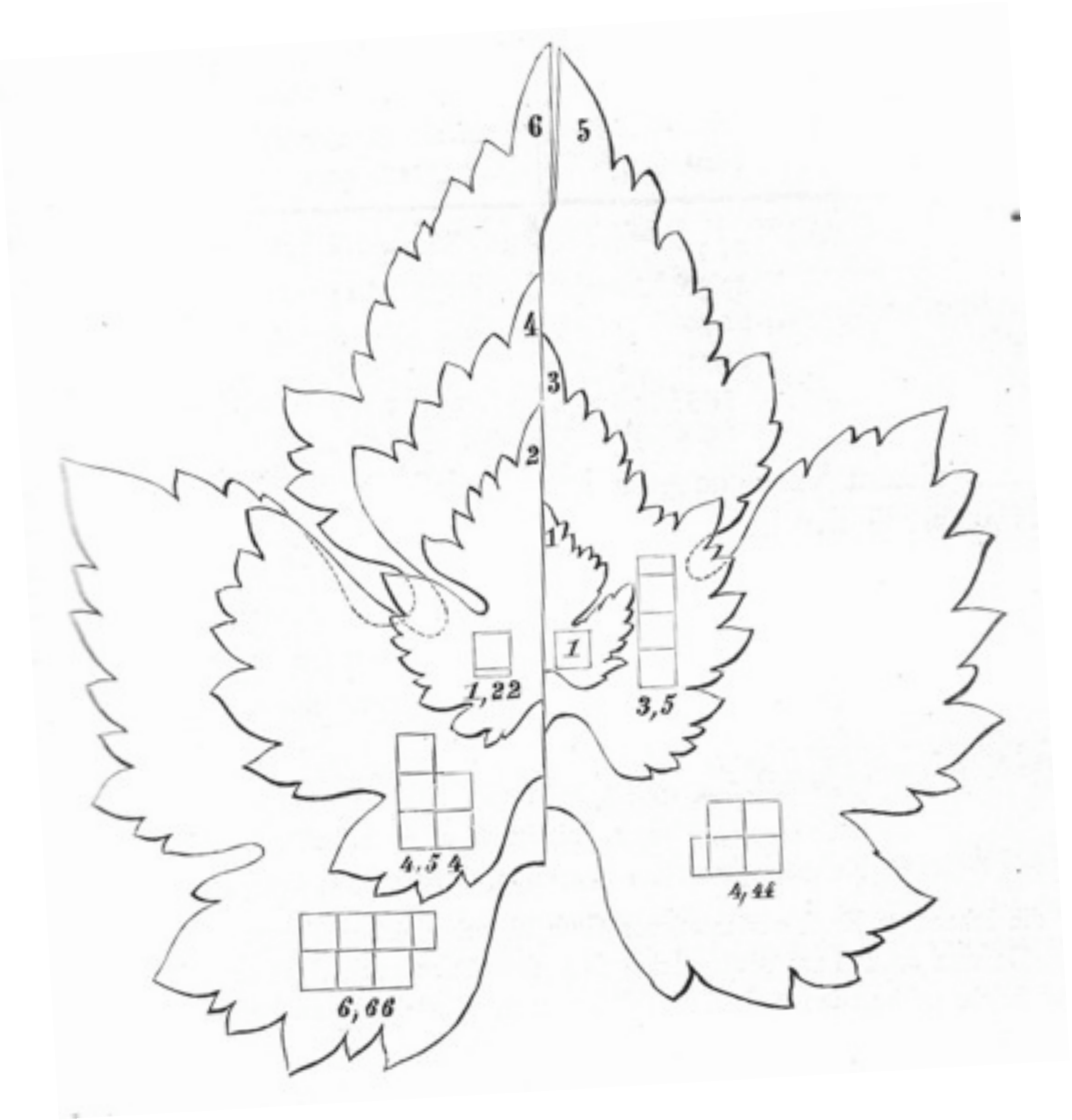

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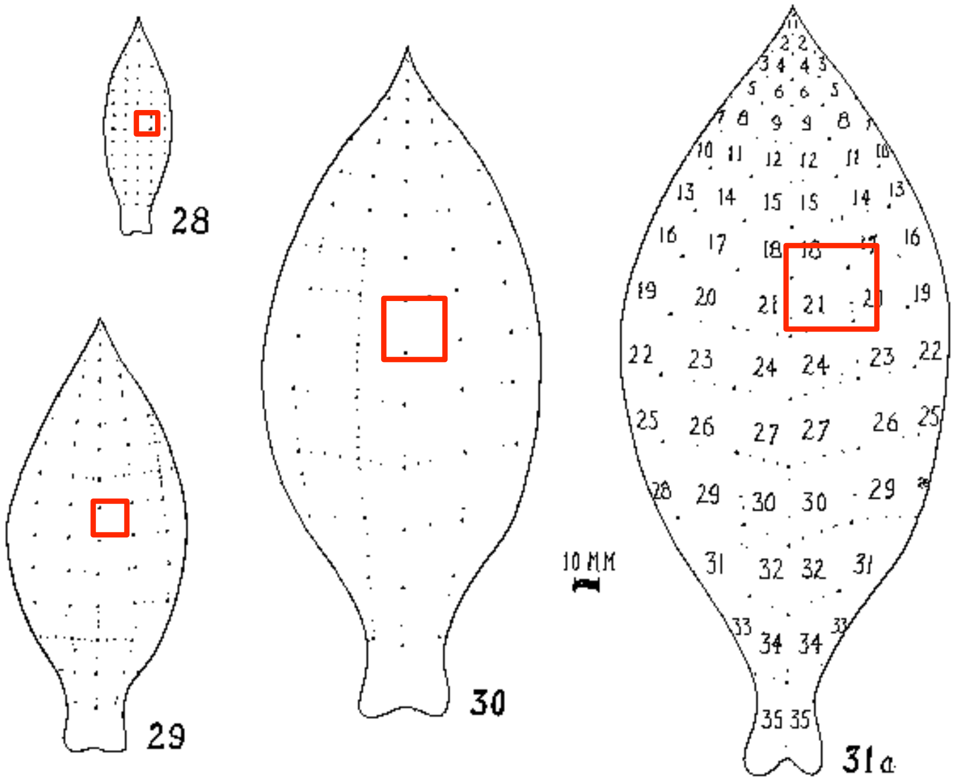

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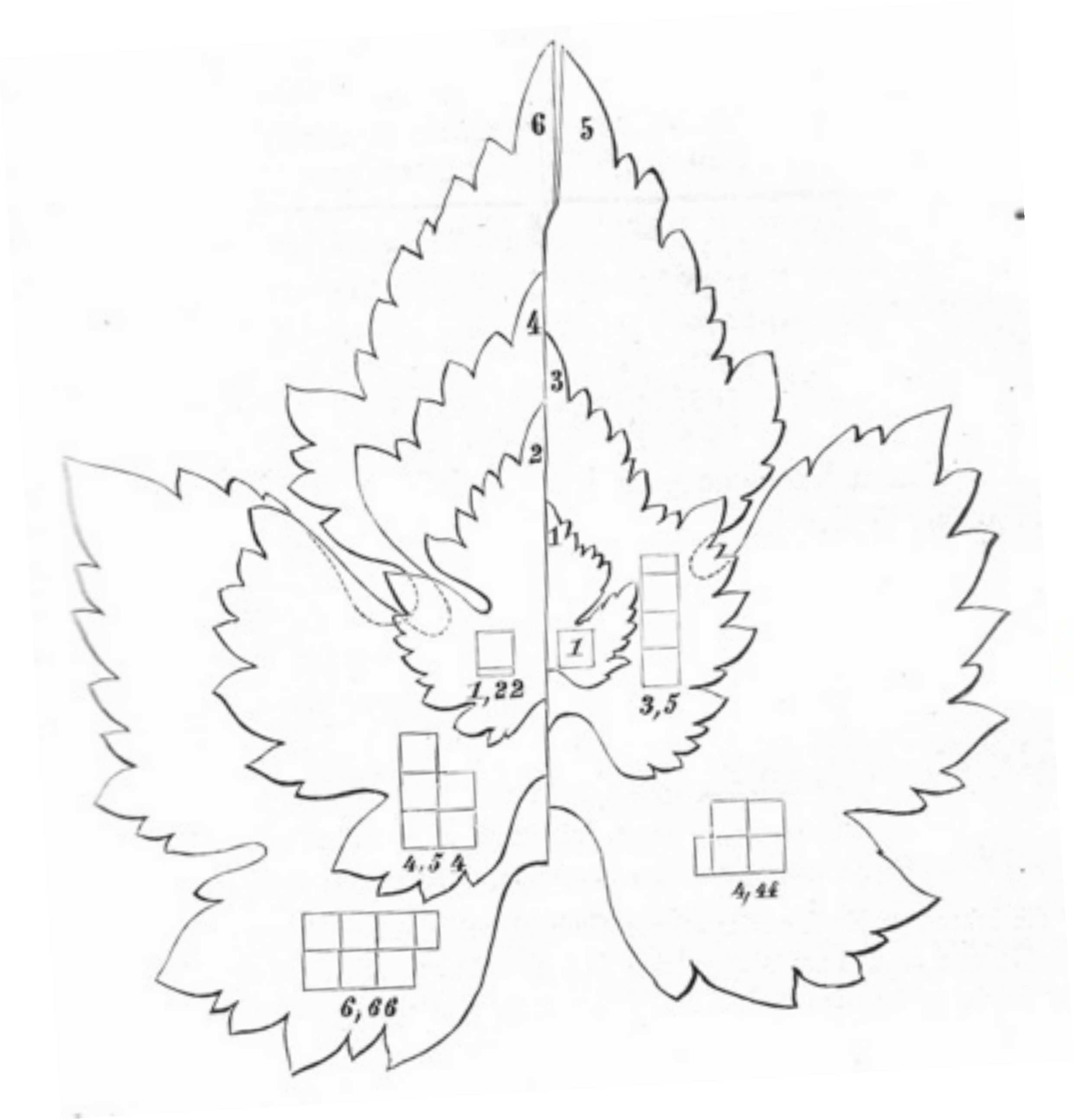
Growth as a genetic problem



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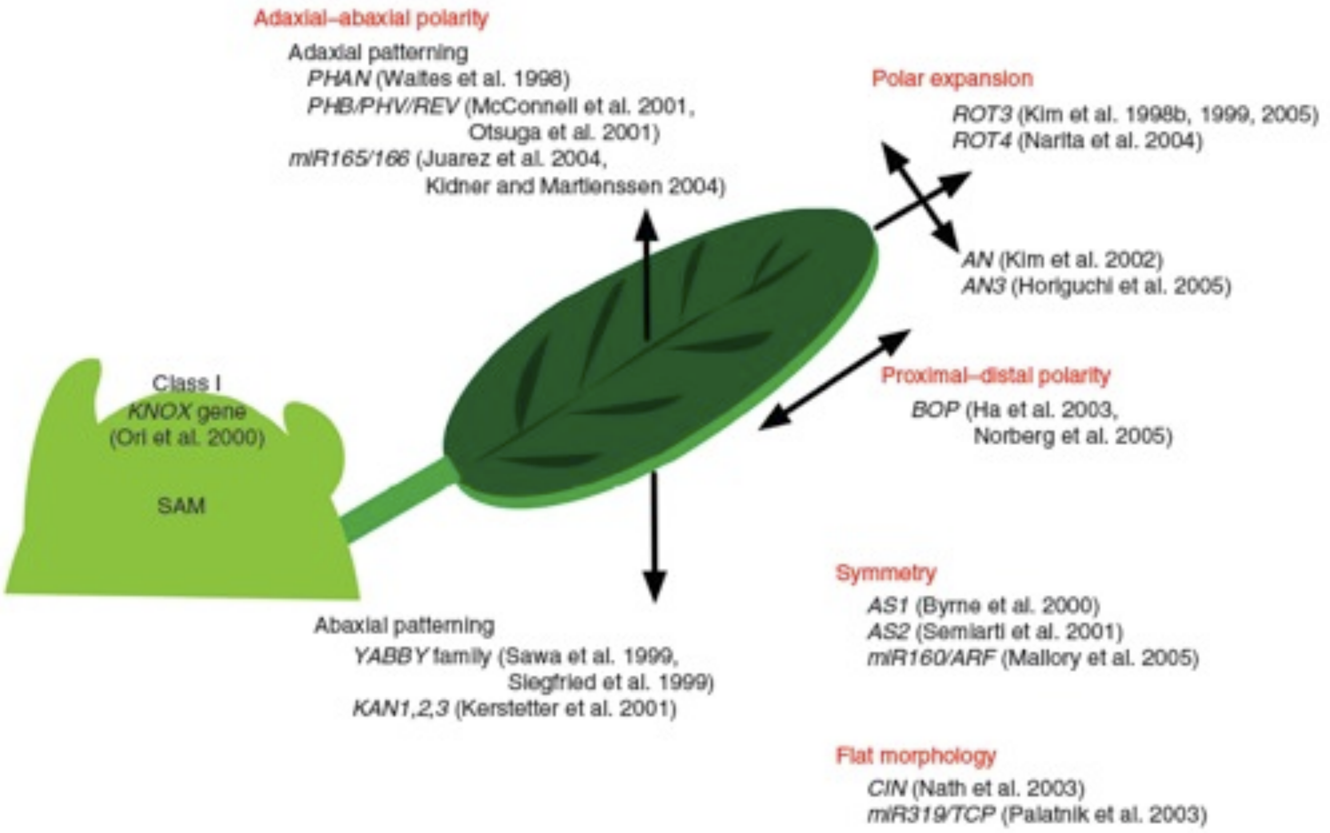


Kim, Cho '06

Growth as a mechanical problem

Auxanometer

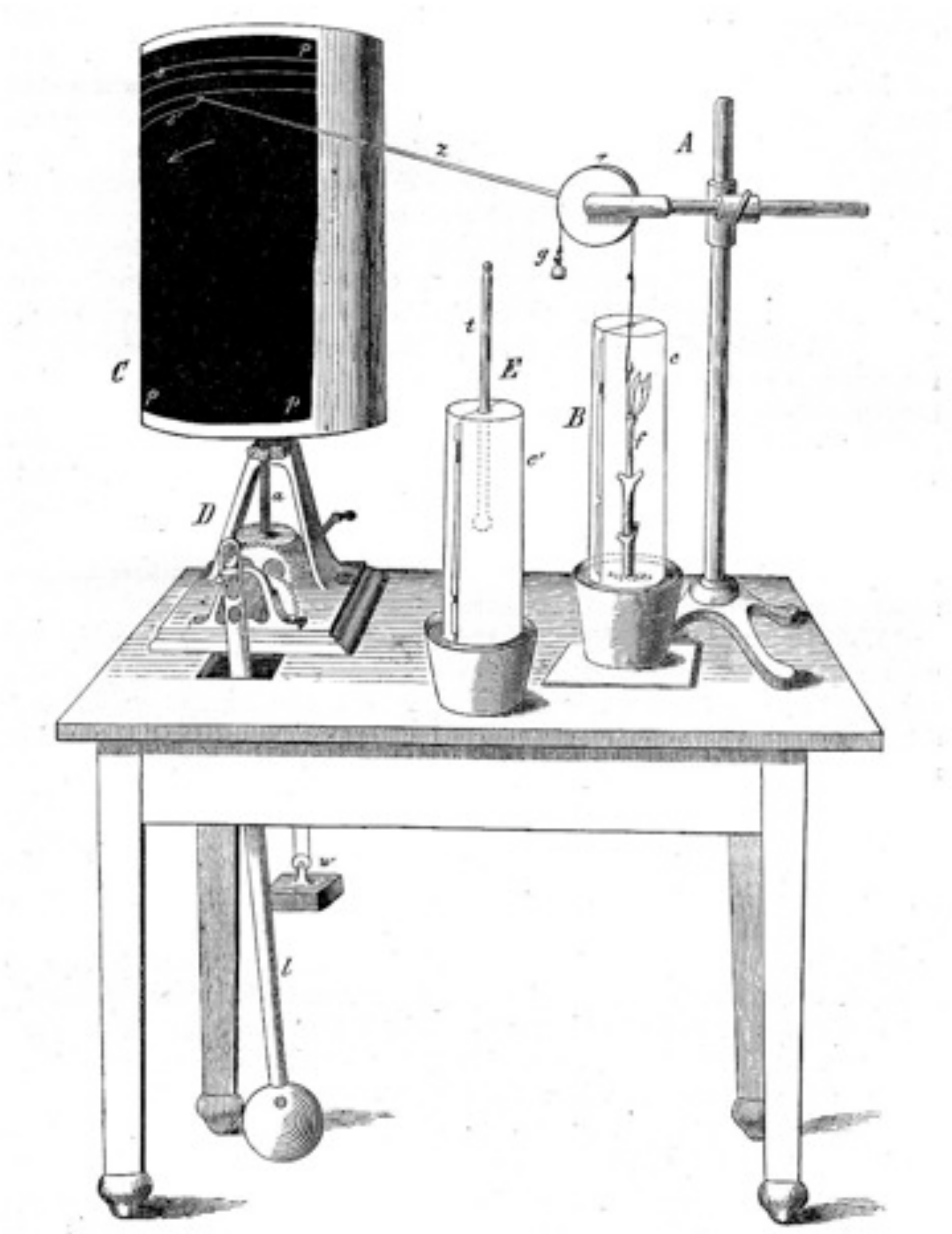
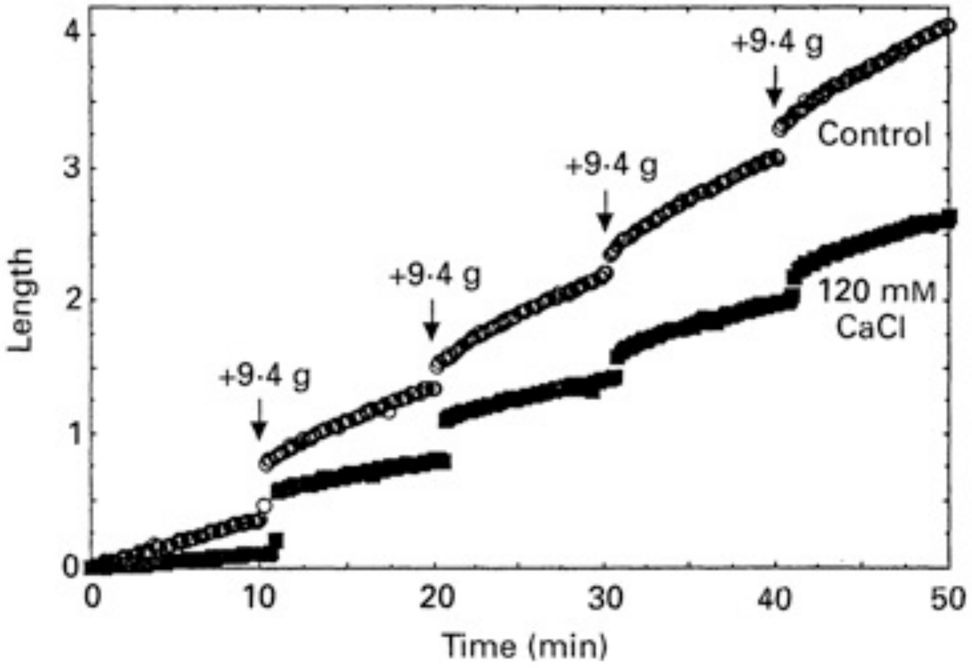


FIG. 348.—The self-recording Auxanometer in its original form.

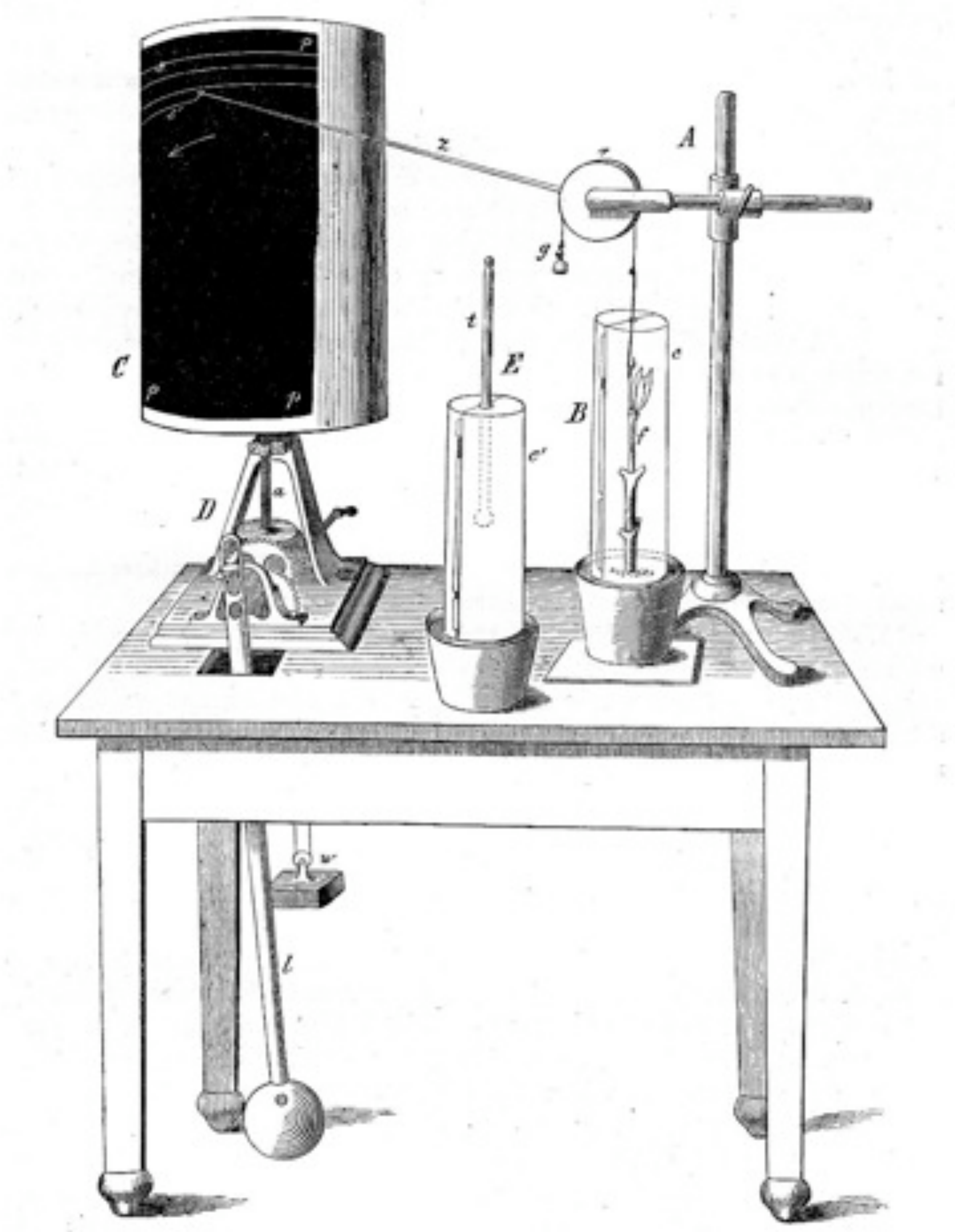
Sachs,
Physiology of Plants 1887



Cramer-Bowman 1991
Maize leaves.

Growth as a mechanical problem

Auxanometer



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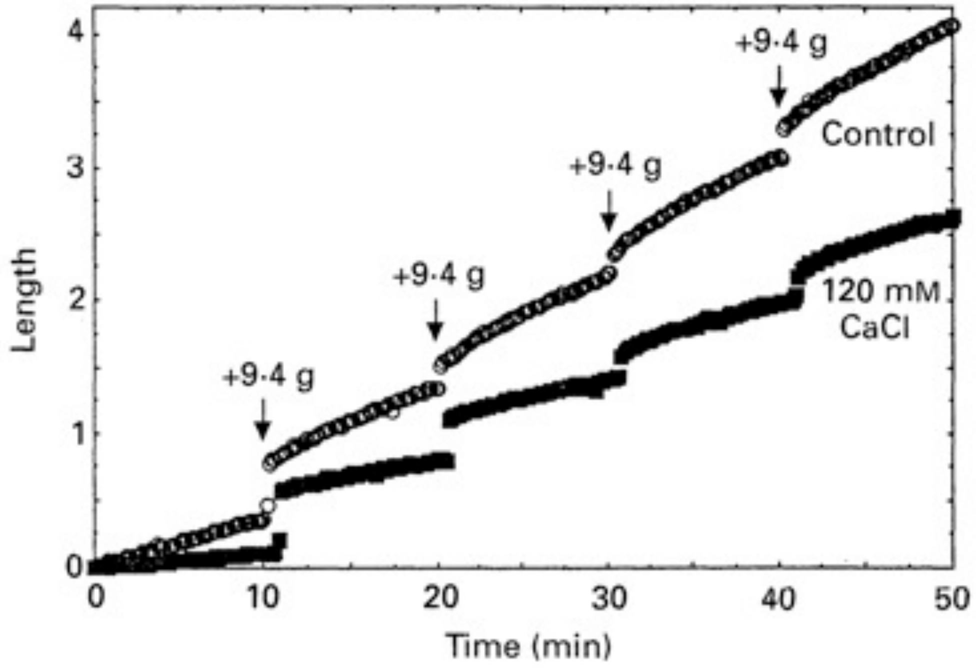

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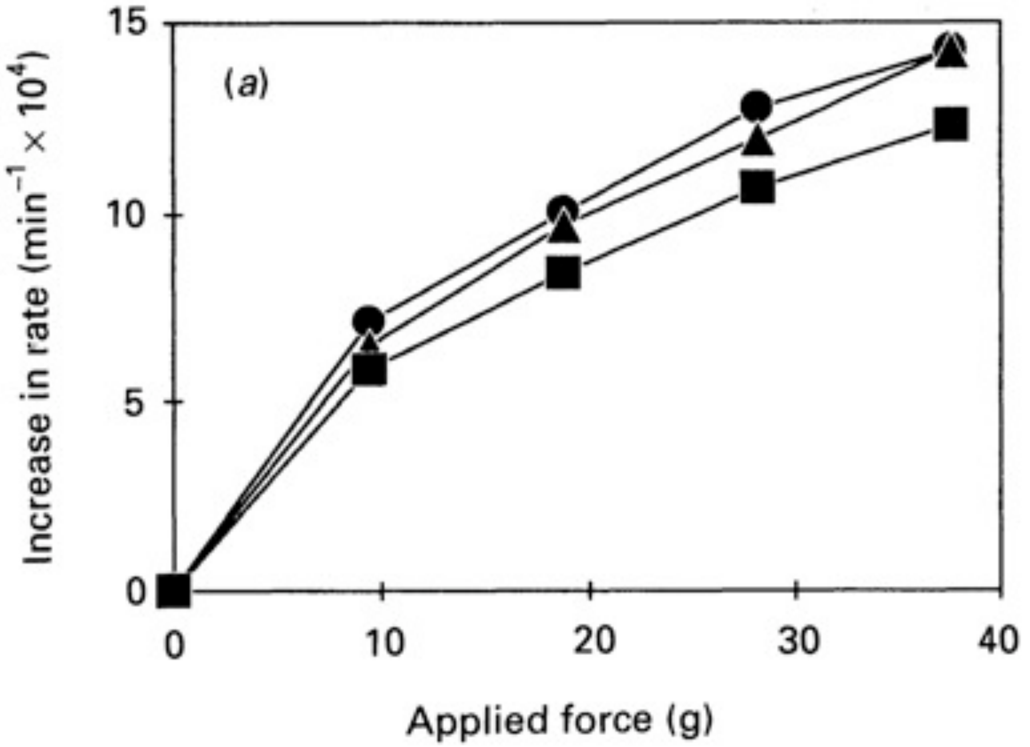
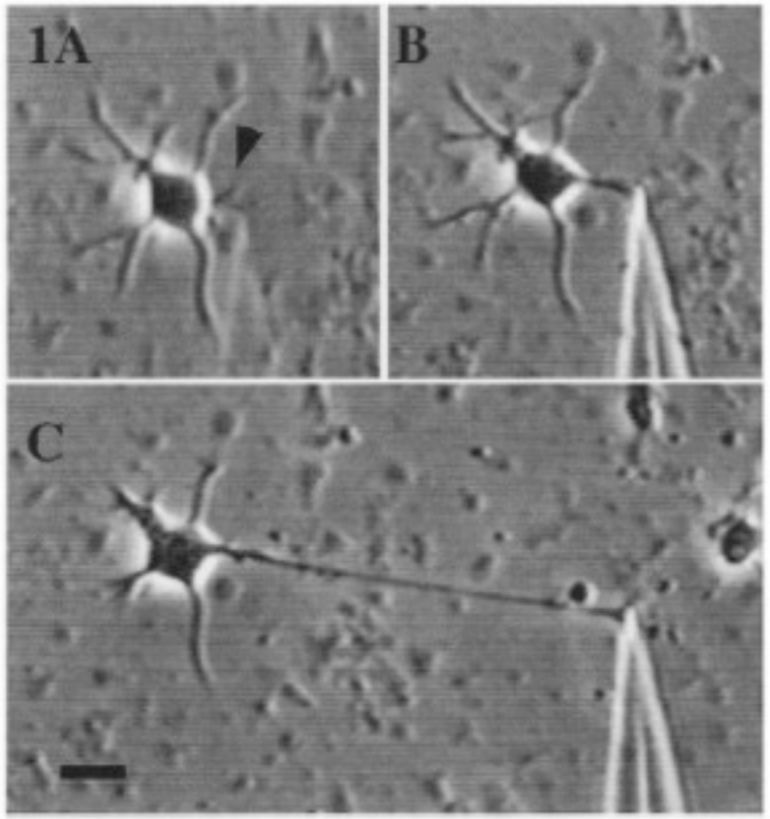


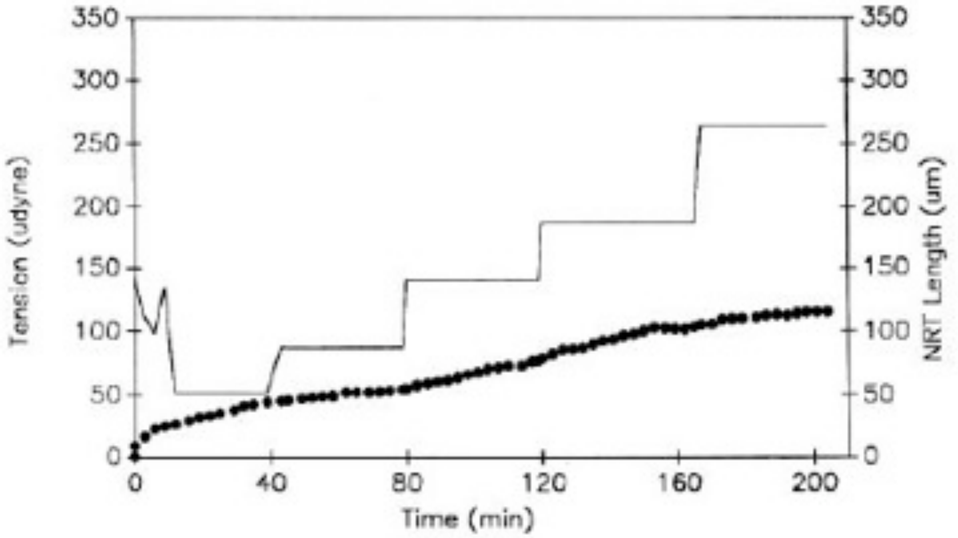
$$\frac{\dot{G}}{G} = f(T)$$

Stress ⇒ Growth: neurons

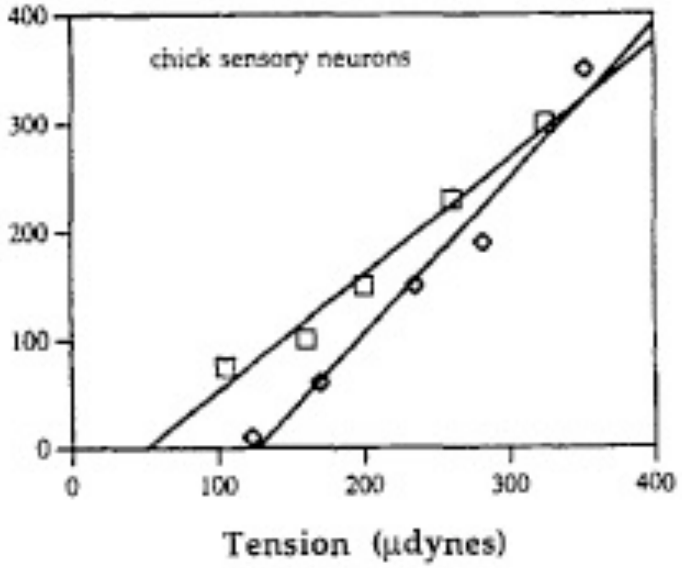
- Pulling on neurons



Growth from applied tension



Growth rate



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Ref: Lamoureux, Heidemann, Buxbaum '89, '90, '91, '97, '02

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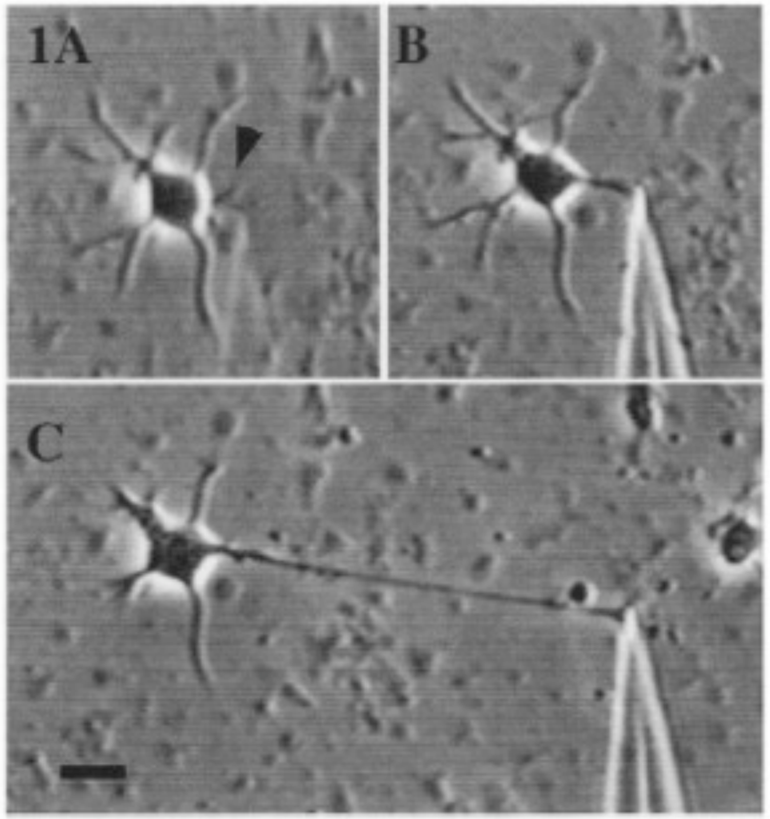

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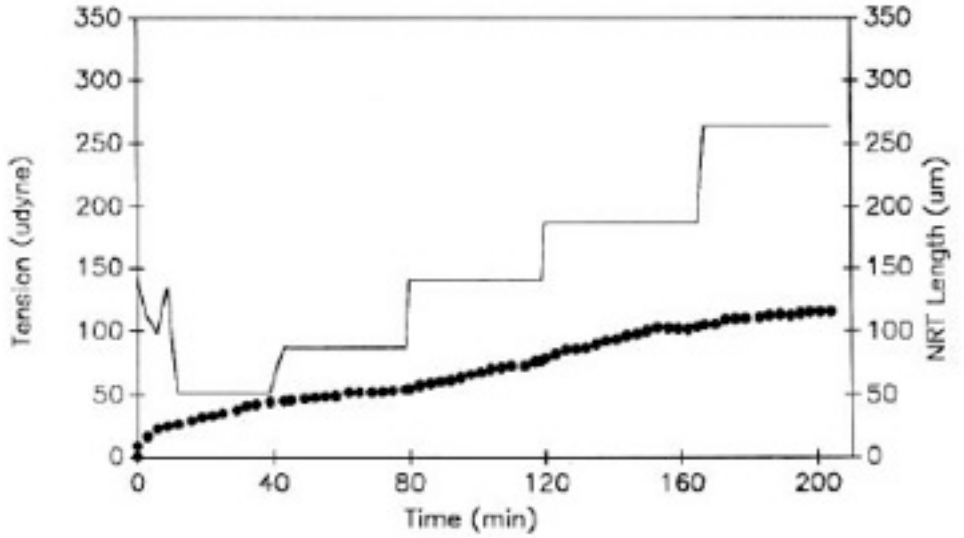

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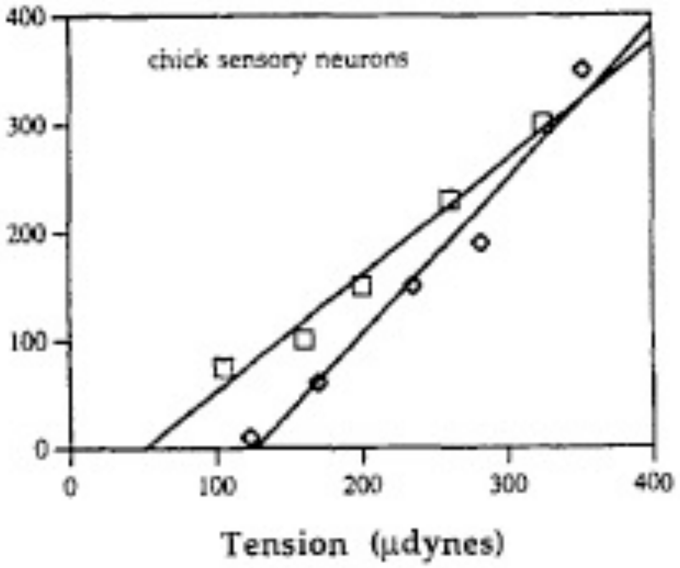
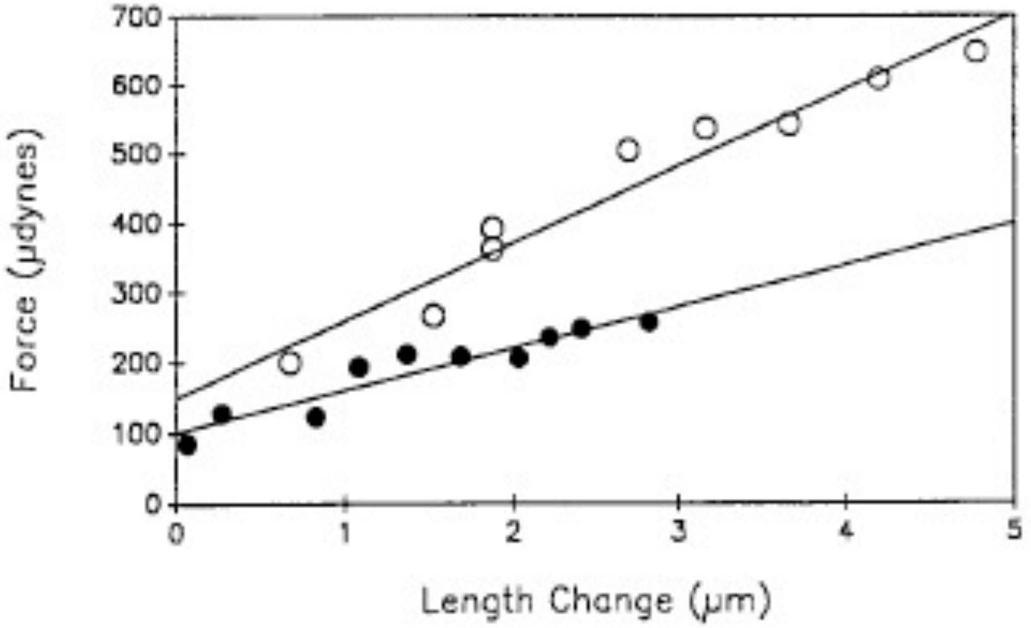


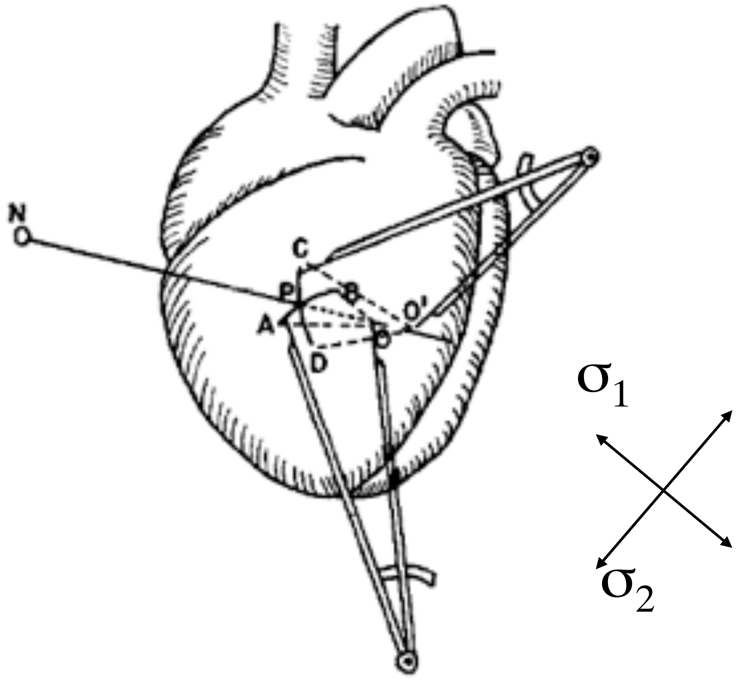
$$\frac{\dot{G}}{G} = f(T)$$

Elasticity



Stress ⇒ Growth: heart

- Mechanisms (“Woods Law” 1892)



Burton, 1957

Stress ⇒ Growth: heart

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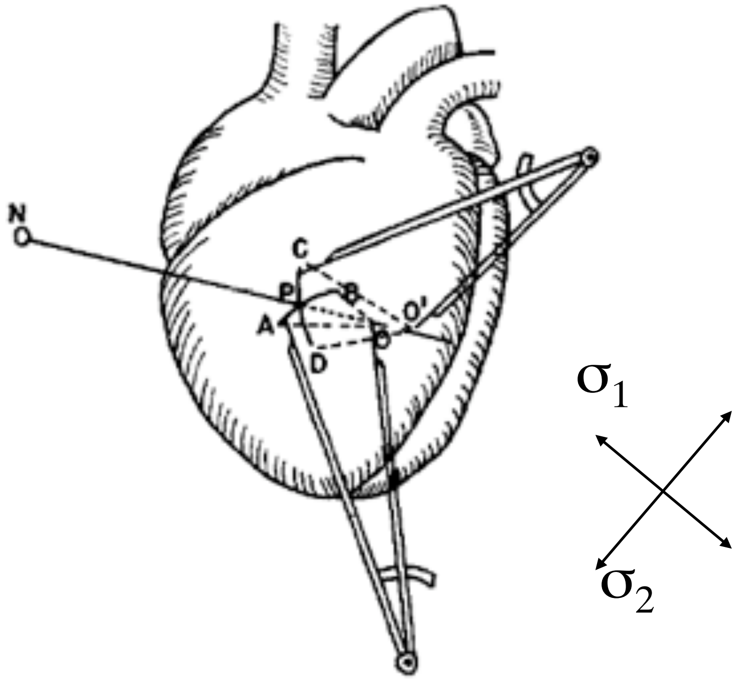
  - Young-Laplace's law



Burton, 1957

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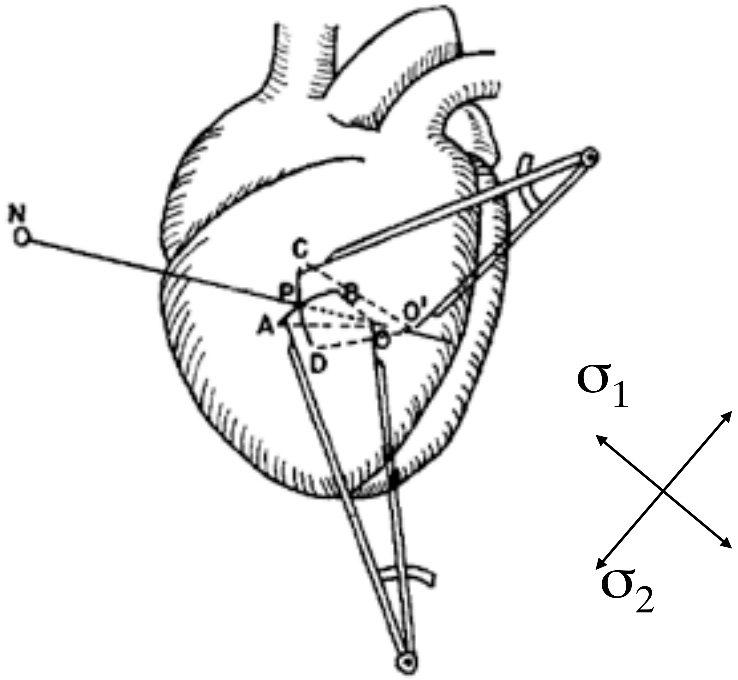

Burton, 1957

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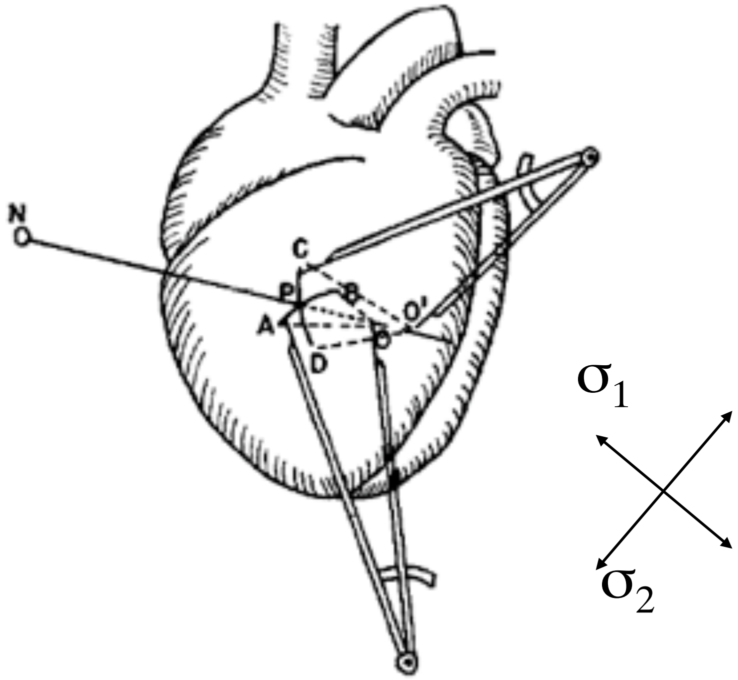
  - Assume isotropy of wall stresses $\sigma_1=\sigma_2$



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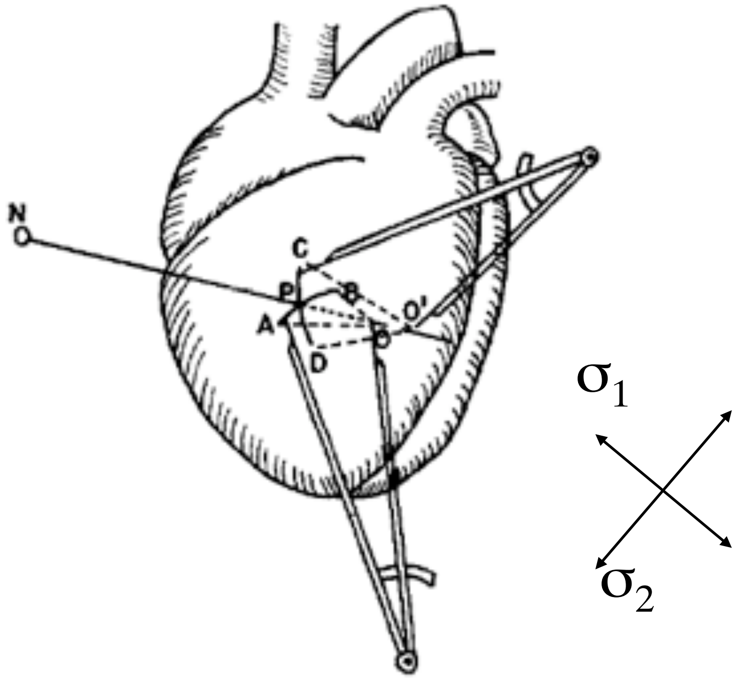
  - Assume isotropy of wall stresses $\sigma_1=\sigma_2$

$$P=\sigma h(1/R_1+1/R_2)$$



Burton, 1957

Stress ⇒ Growth: heart

- Mechanisms (“Woods Law” 1892)
  - ✤ Young-Laplace’s law

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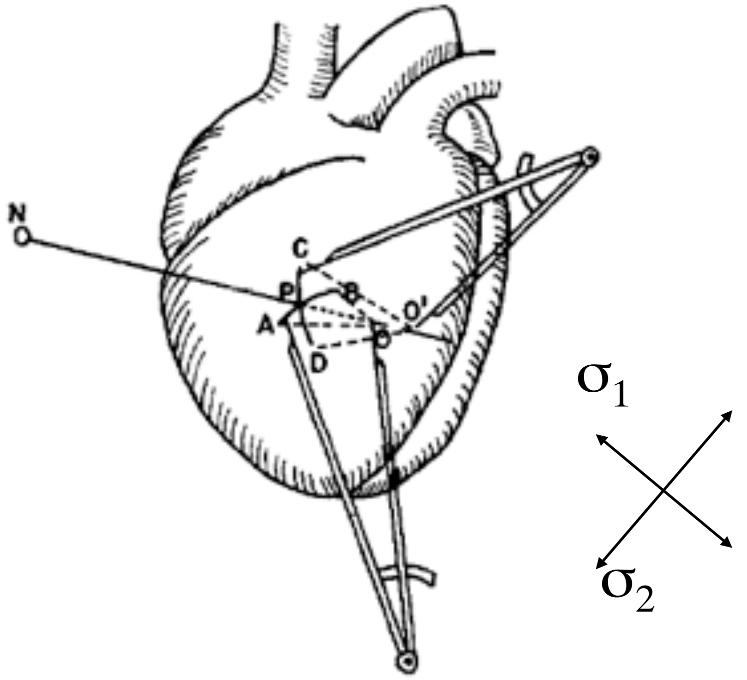
$$P=\sigma h(1/R_1+1/R_2)$$

  - ✤ C=$P/\sigma=h(1/R_1+1/R_2)$ is nearly constant



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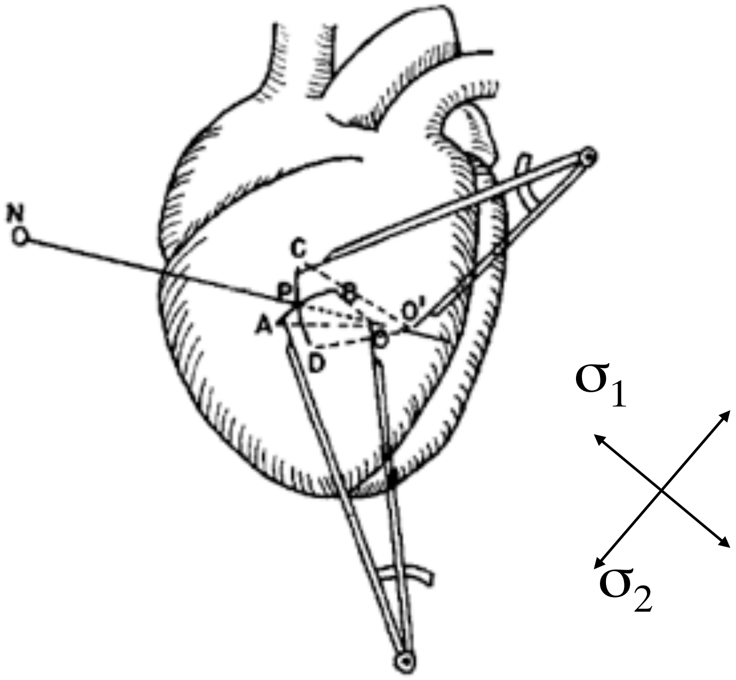
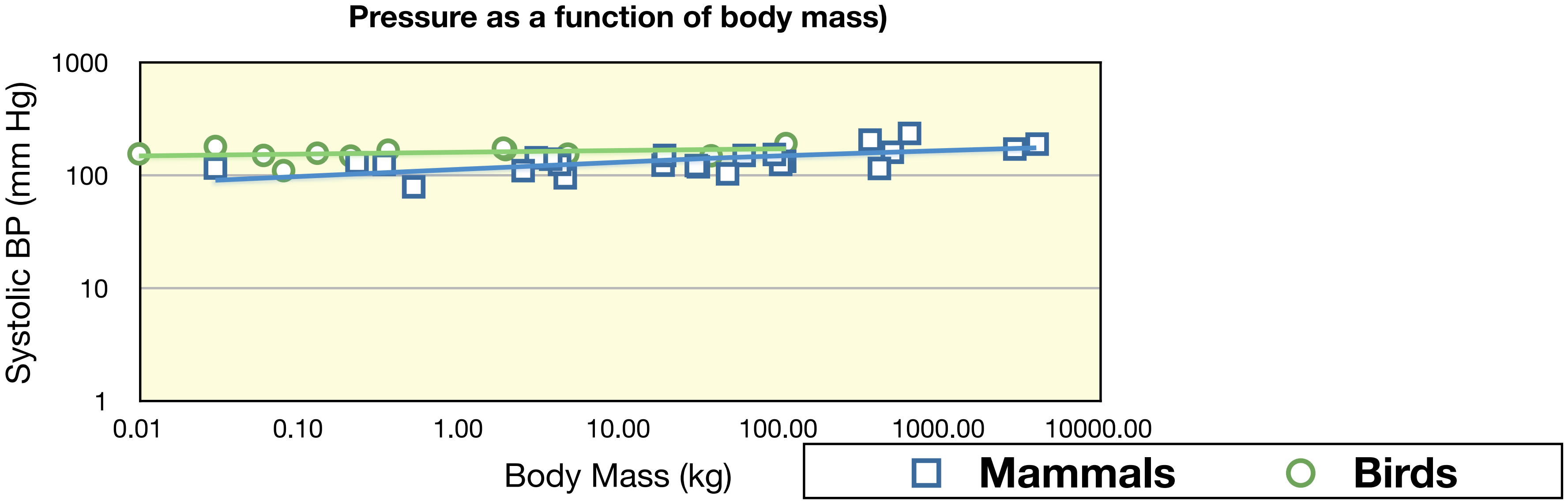


Burton, 1957



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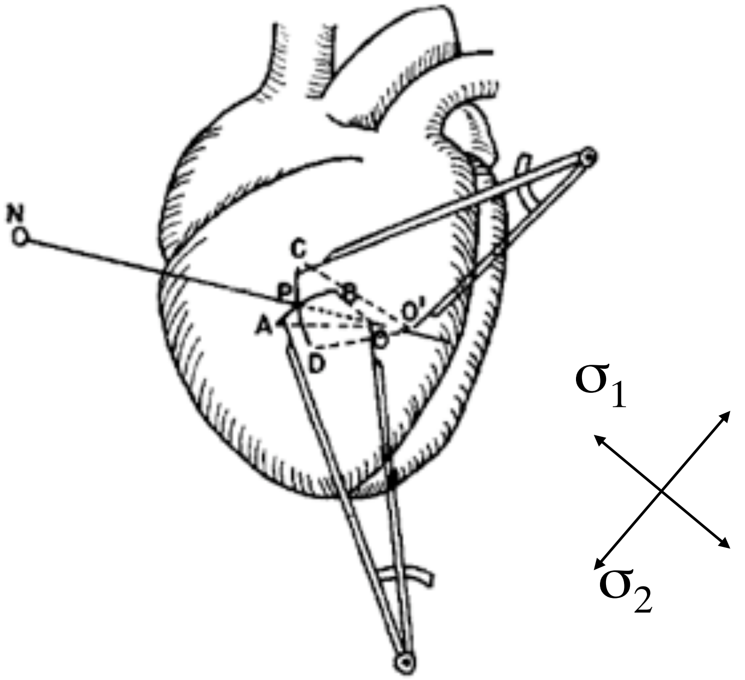
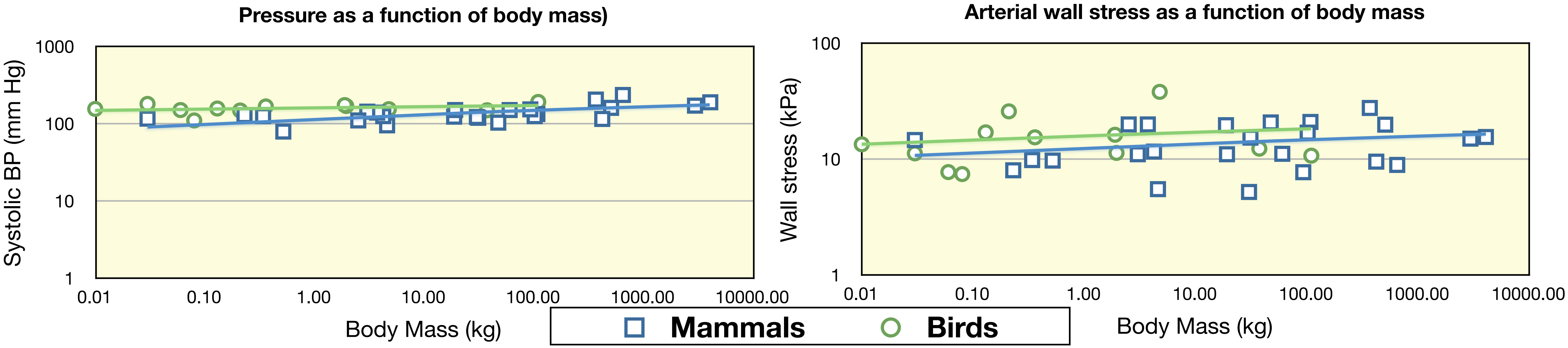


Burton, 1957


Pressure as a function of body mass)

Arterial wall stress as a function of body mass

Stress ⇒ Growth: heart

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  - Young-Laplace's law

$$P=h(\sigma_1/R_1+\sigma_2/R_2)$$

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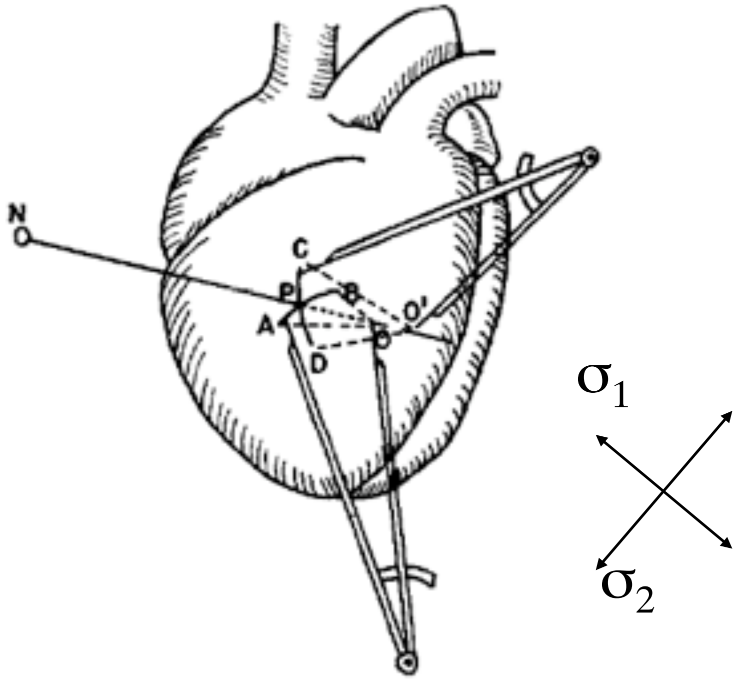
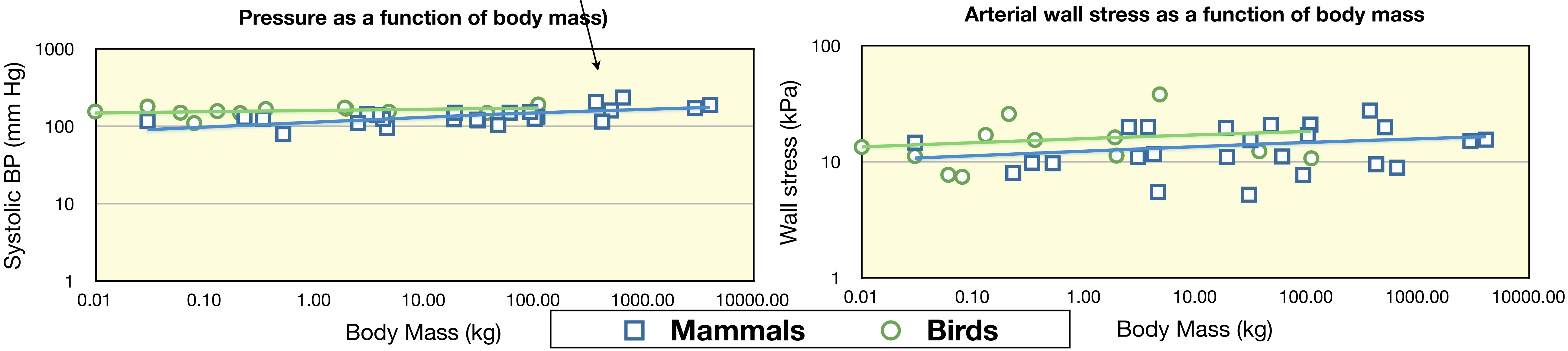


Burton, 1957



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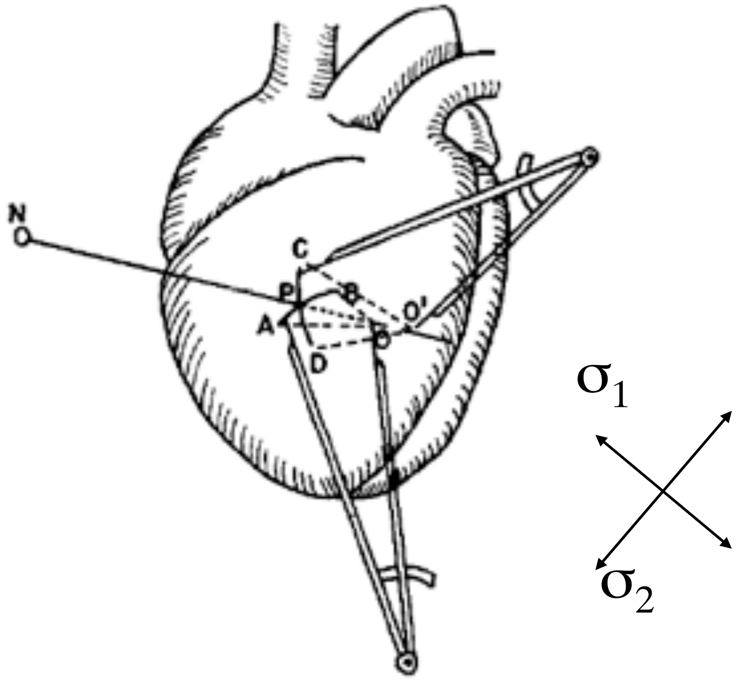
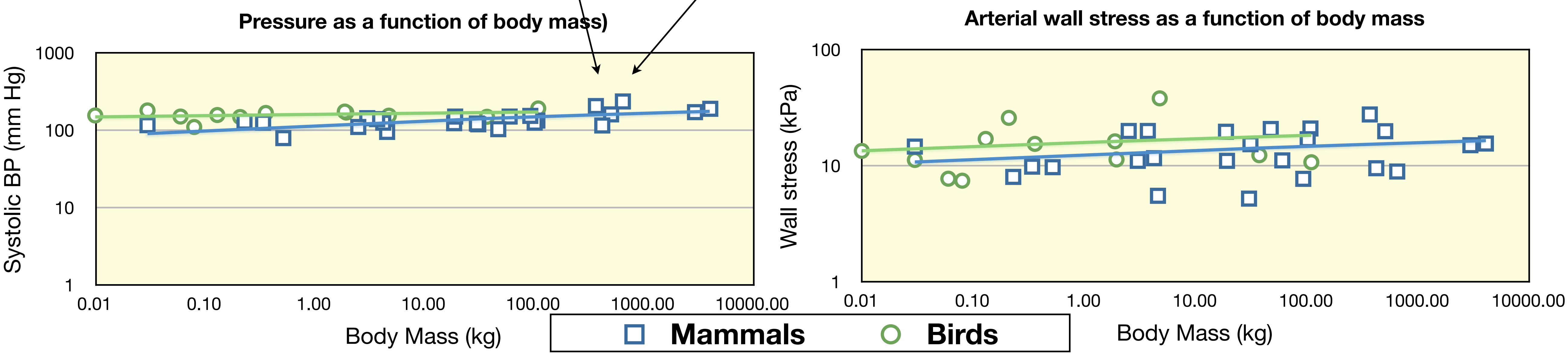


Burton, 1957



Pressure as a function of body mass)

Arterial wall stress as a function of body mass

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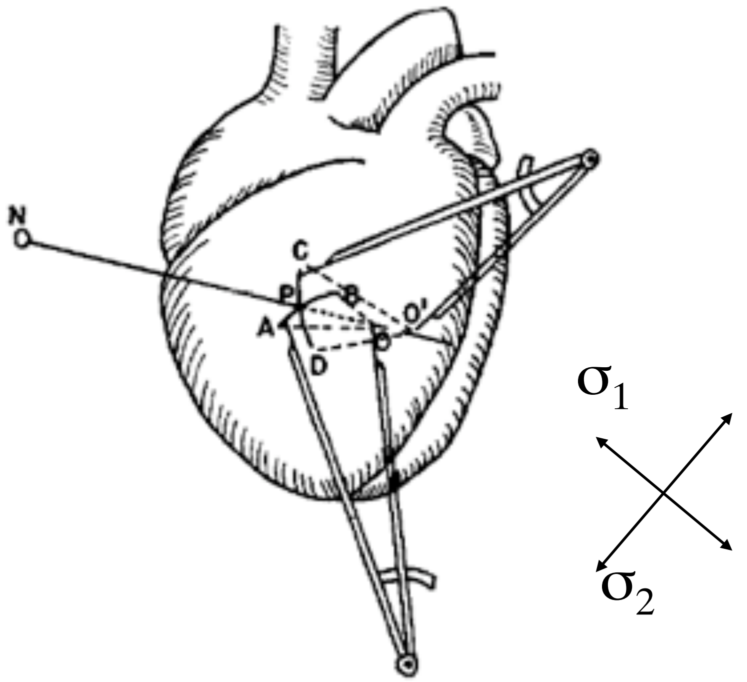
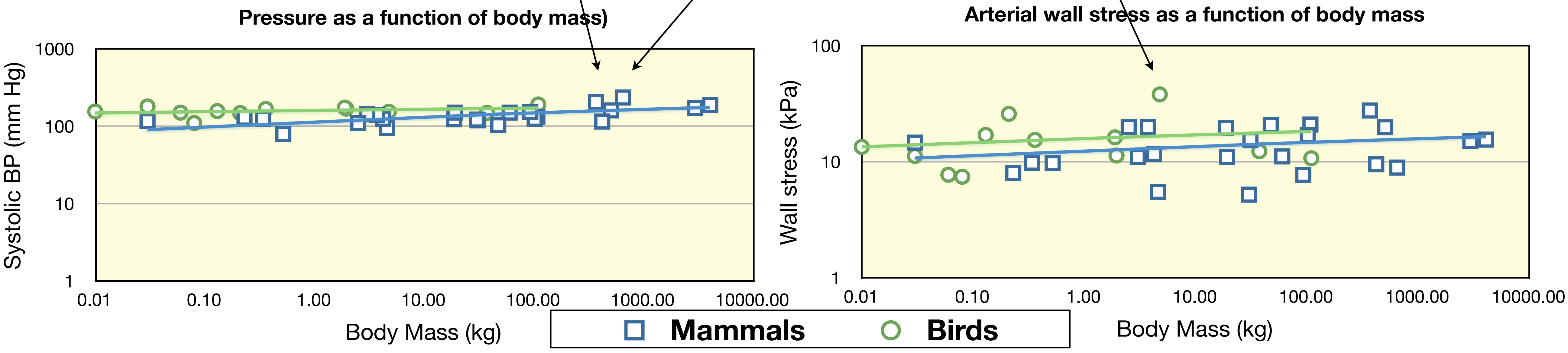
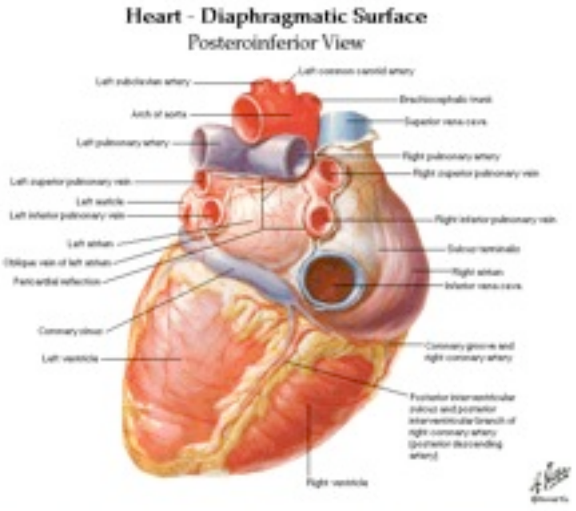


Burton, 1957



Stress ⇒ Growth: heart

- Stimuli for growth and remodeling



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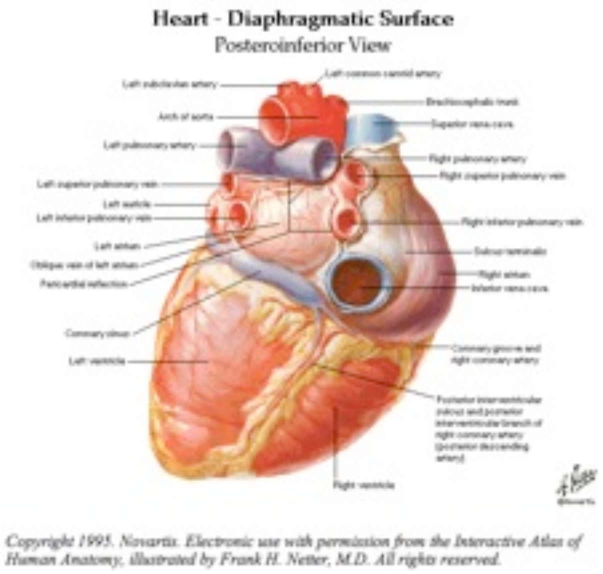
Stress ⇒ Growth: heart

- Stimuli for growth and remodeling
    - Growth to keep wall stress constant

$$\sigma \sim 2\,PR/h$$



Heart - Diaphragmatic Surface
Posteroinferior View

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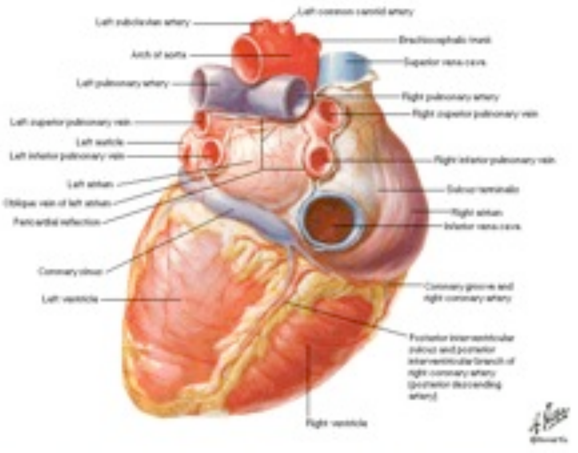
Stress ⇒ Growth: heart

- Stimuli for growth and remodeling
  - Growth to keep wall stress constant

$$\sigma \sim 2\,PR/h$$

  - P increases (pressure overload) → h increases (thickening)


Heart - Diaphragmatic Surface
Posteroinferior View

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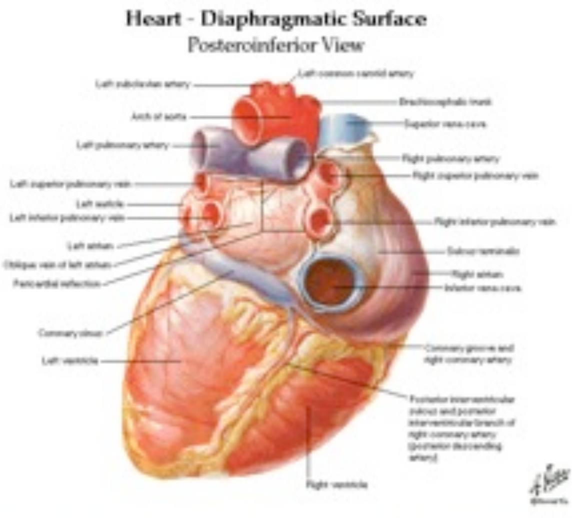
Stress ⇒ Growth: heart

- Stimuli for growth and remodeling
  - Growth to keep wall stress constant

$$\sigma \sim 2\, PR/h$$

  - P increases (pressure overload) → h increases (thickening)
  - R increases (volume overload) → h increases (thickening)



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- Stimuli for growth and remodeling
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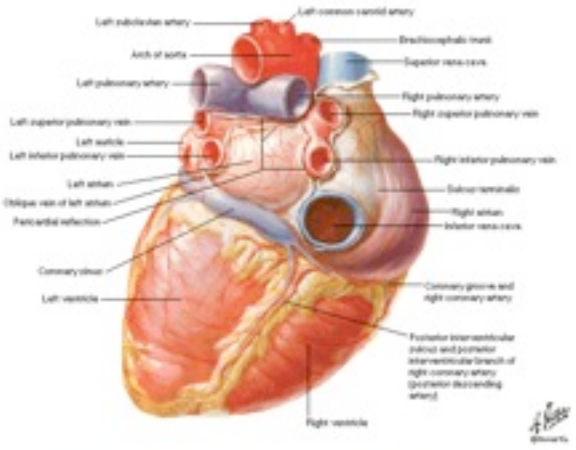
  - P increases (pressure overload) → h increases (thickening)
  - R increases (volume overload) → h increases (thickening)
- Notion of homeostatic stress

Heart - Diaphragmatic Surface
Posteroinferior View



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Stress ⇒ Growth: heart

- Stimuli for growth and remodeling
  - Growth to keep wall stress constant

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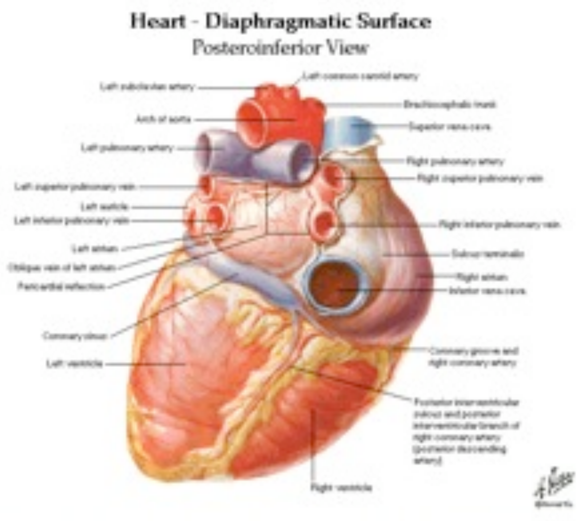
  - R increases (volume overload) → h increases (thickening)
- Notion of homeostatic stress
  - The homeostatic stress $\mathbf{T}^*$ is a preferred (target stress) for growth and remodeling.



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Stress ⇒ Growth: heart

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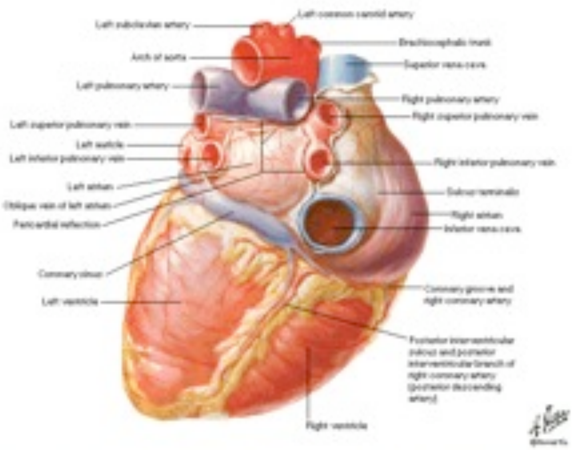
Heart - Diaphragmatic Surface
Posteroinferior View

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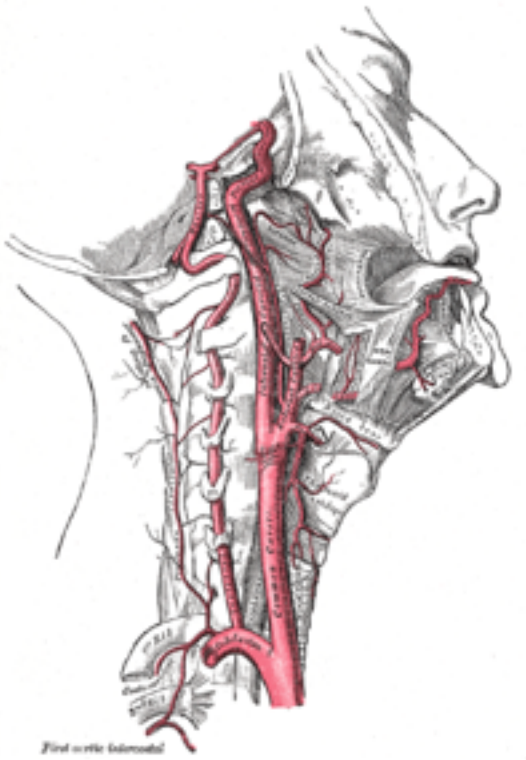
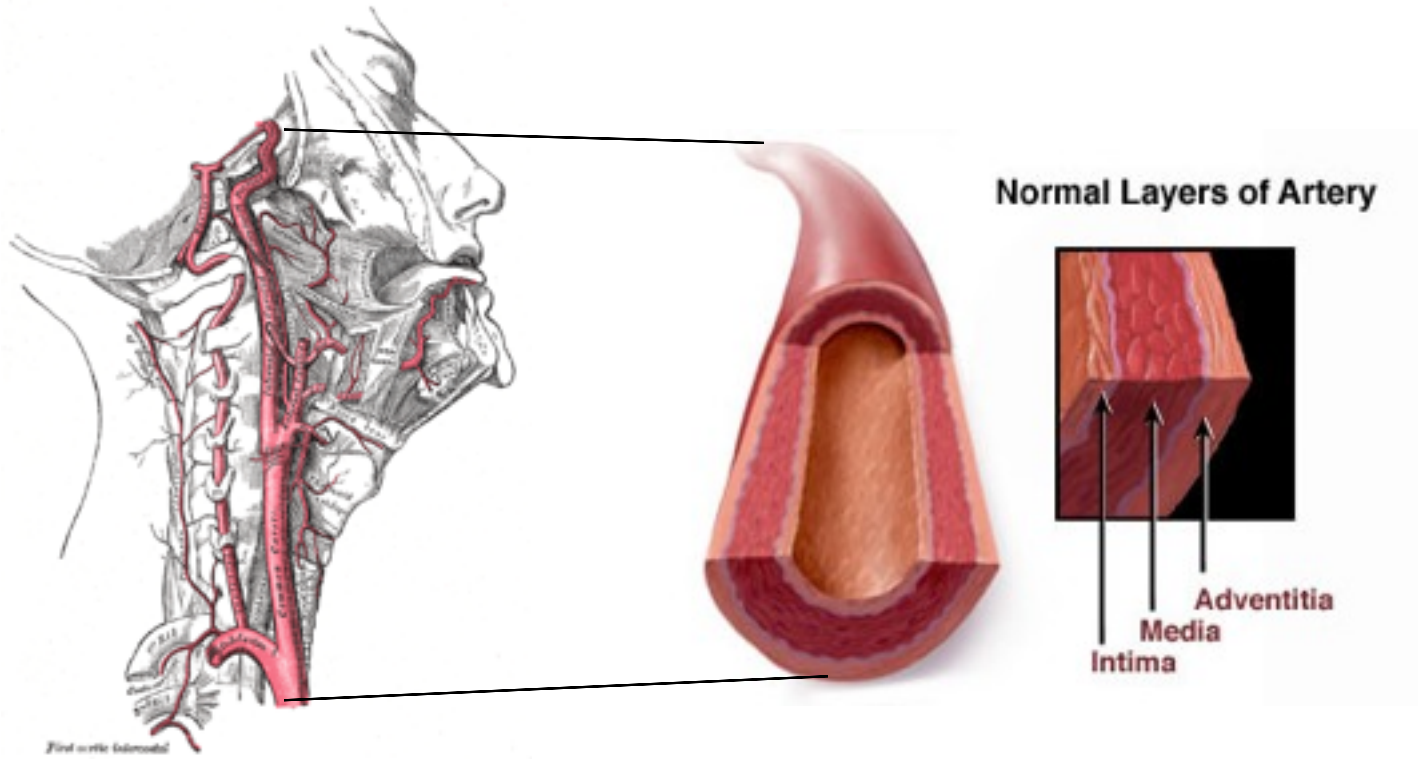
$$\boxed{\frac{\dot{\mathbf{G}}}{\mathbf{G}} = f(\mathbf{T}-\mathbf{T}^*)}$$

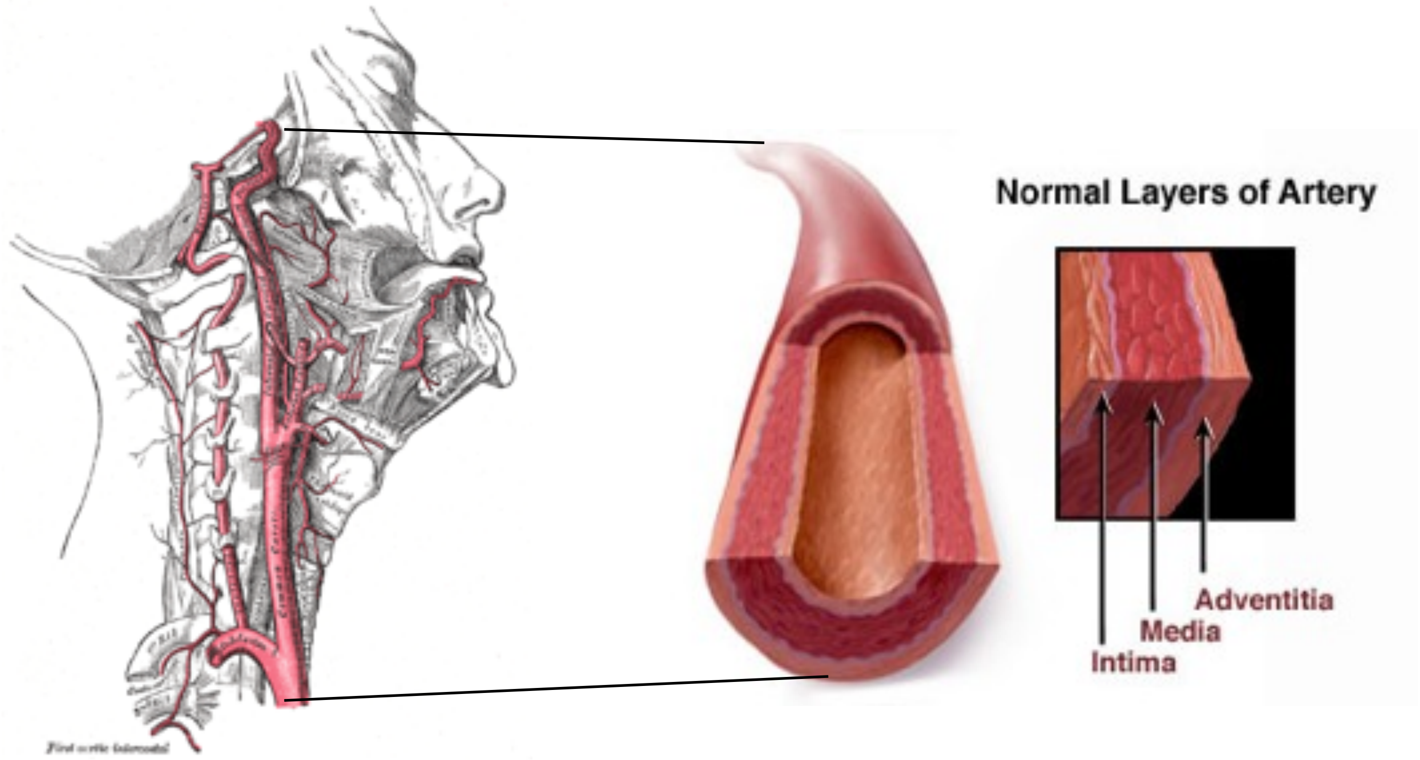
Stress ⇒ Growth: arteries



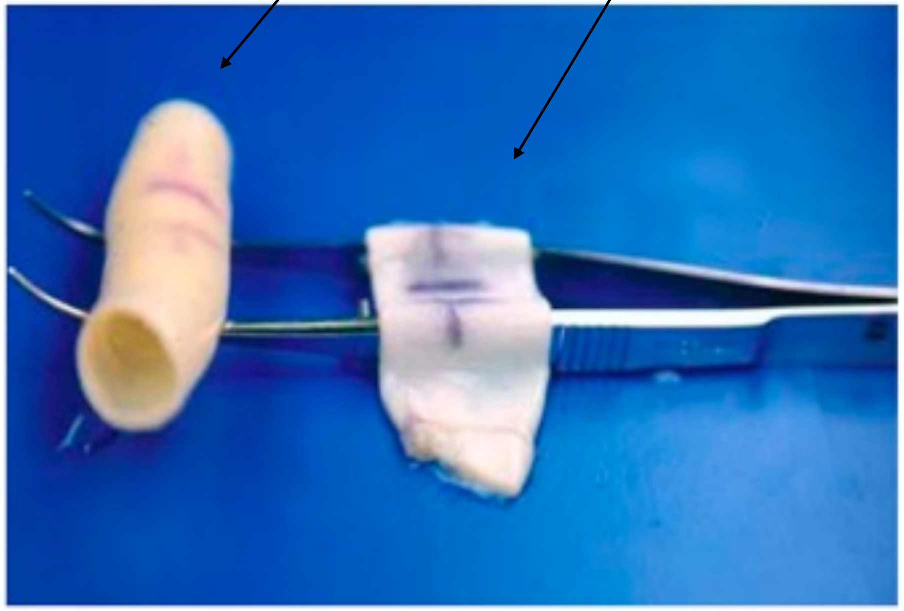
Stress ⇒ Growth: arteries



Stress ⇒ Growth: arteries

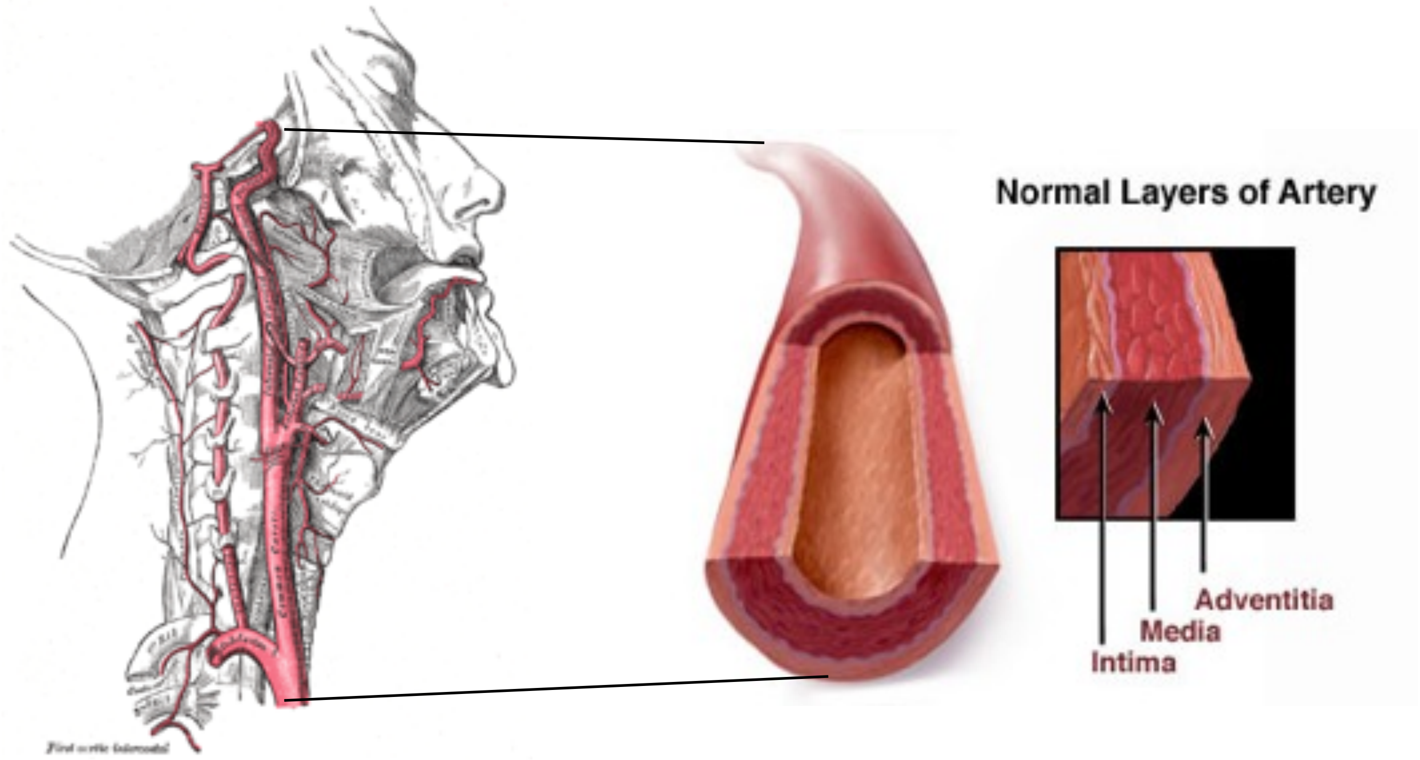


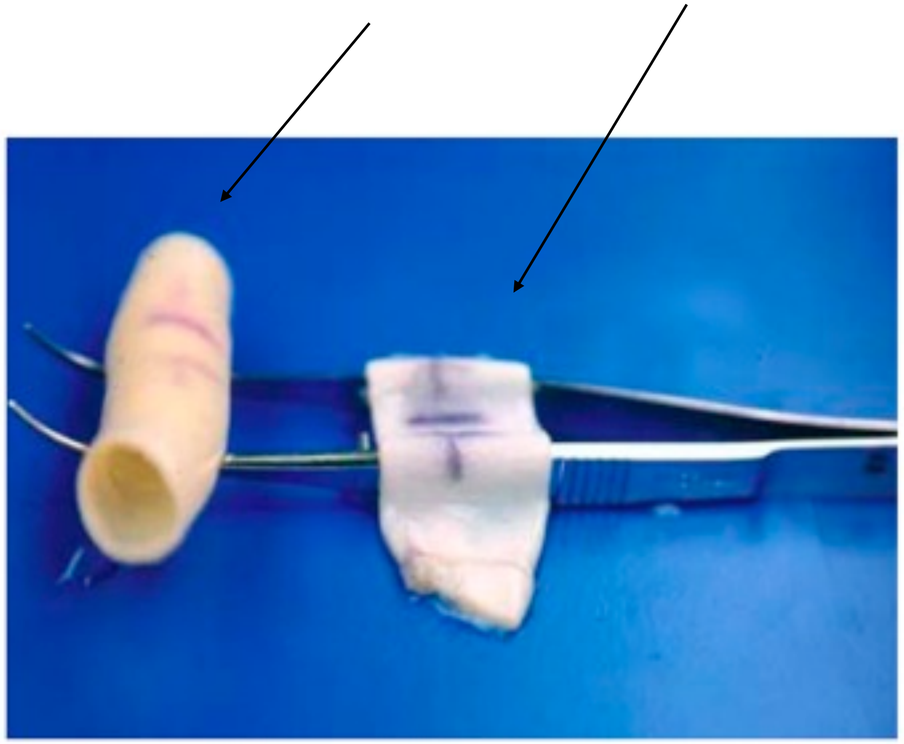
Media-intima and adventitia



Picture from Holzapfel-Ogden

Stress ⇒ Growth: arteries



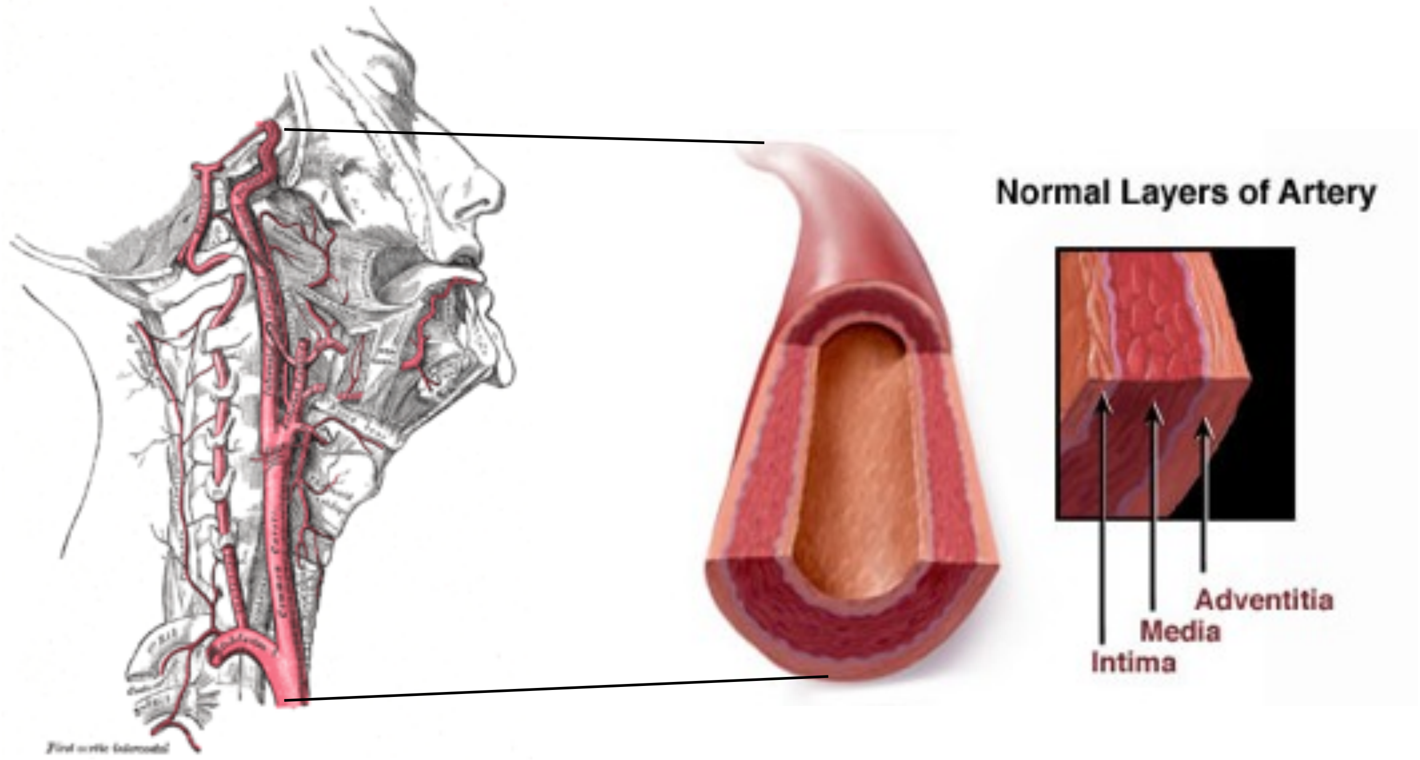


Media-intima and adventitia

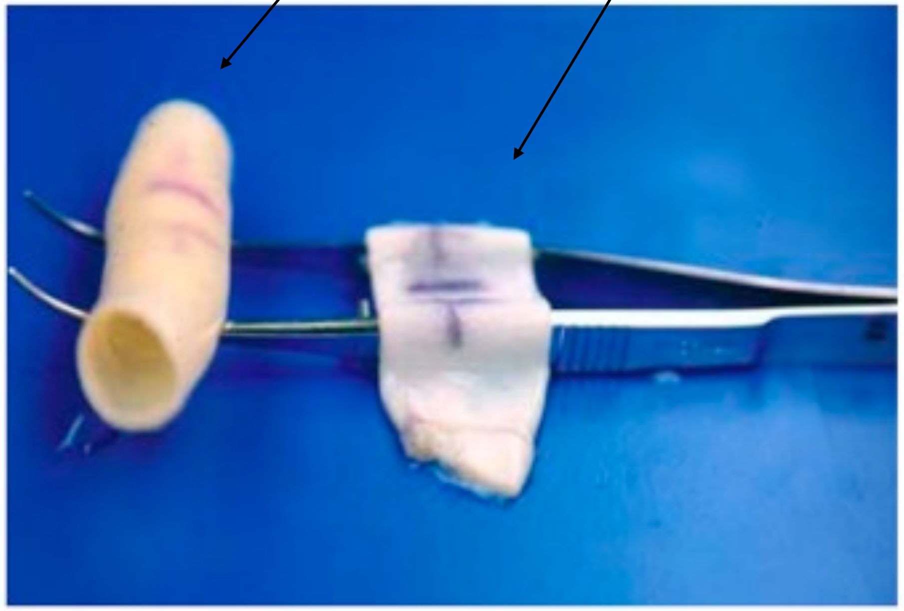
Picture from Holzapfel-Ogden

- Stimuli for growth

Stress ⇒ Growth: arteries



Media-intima and adventitia

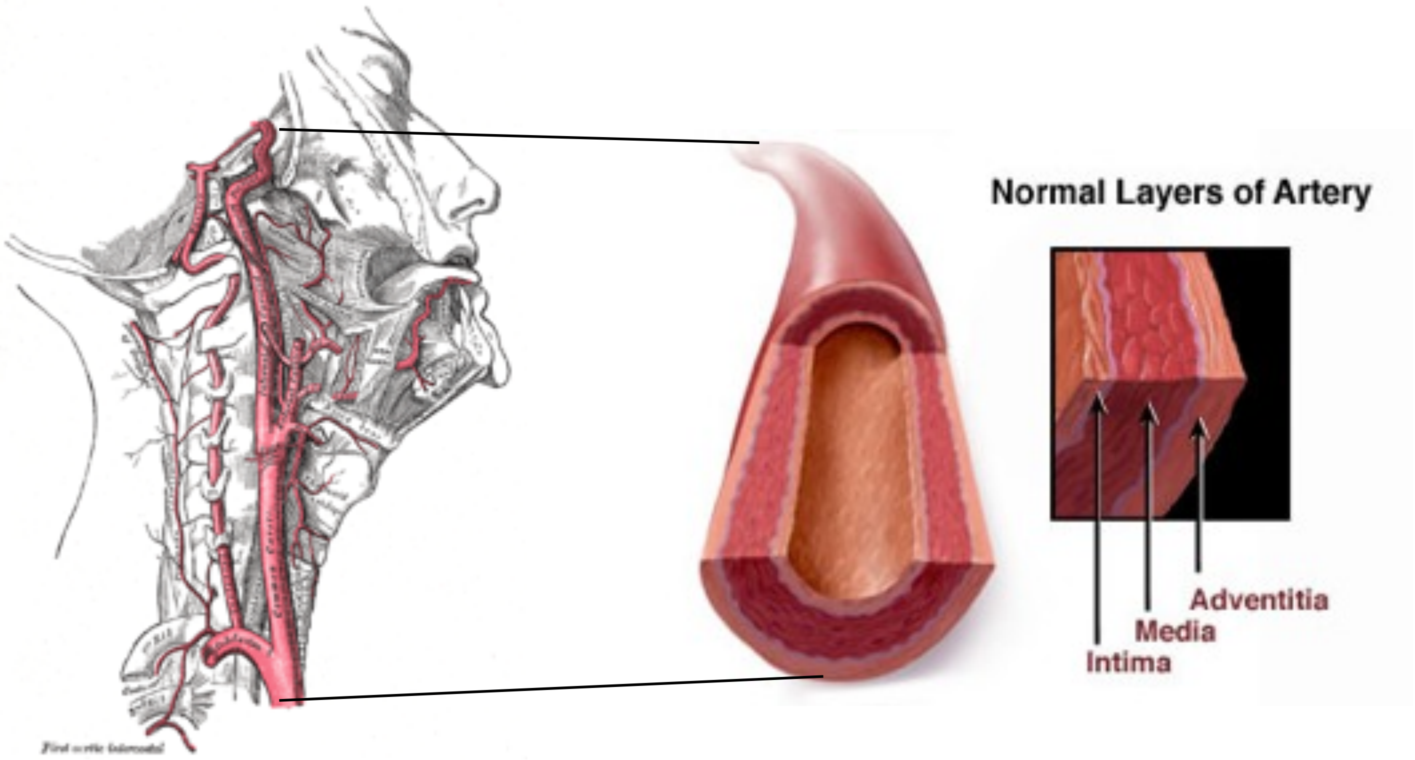


Picture from Holzapfel-Ogden

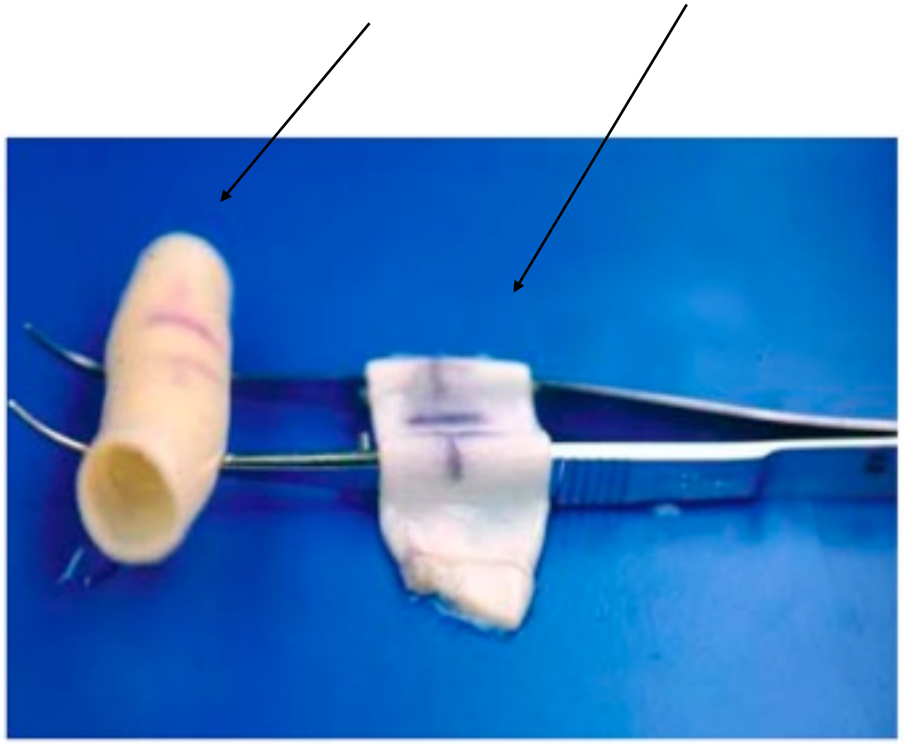
- Stimuli for growth
  - Increase in pressure → Increase in thickness (Thoma 1893)

    Normalize circumferential stress

Stress ⇒ Growth: arteries



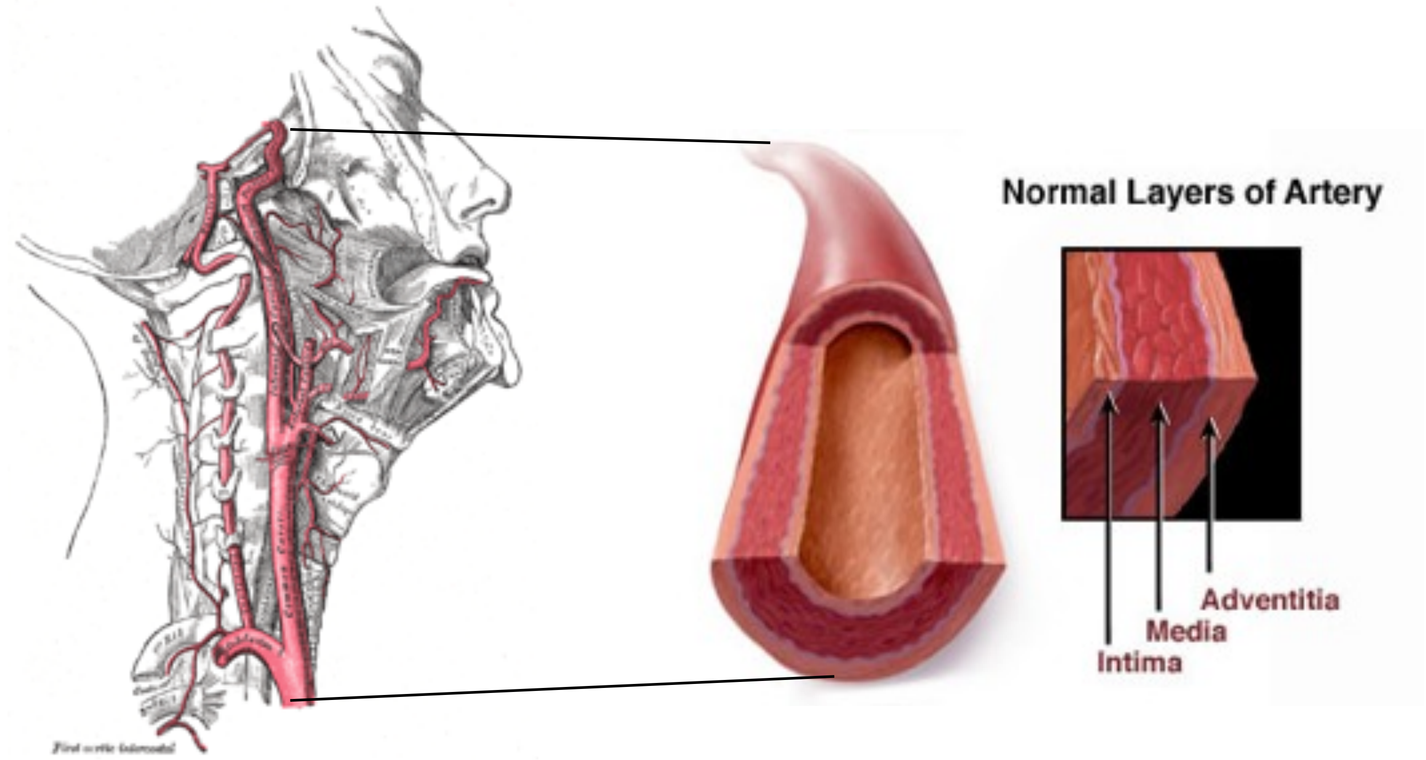
Media-intima and adventitia



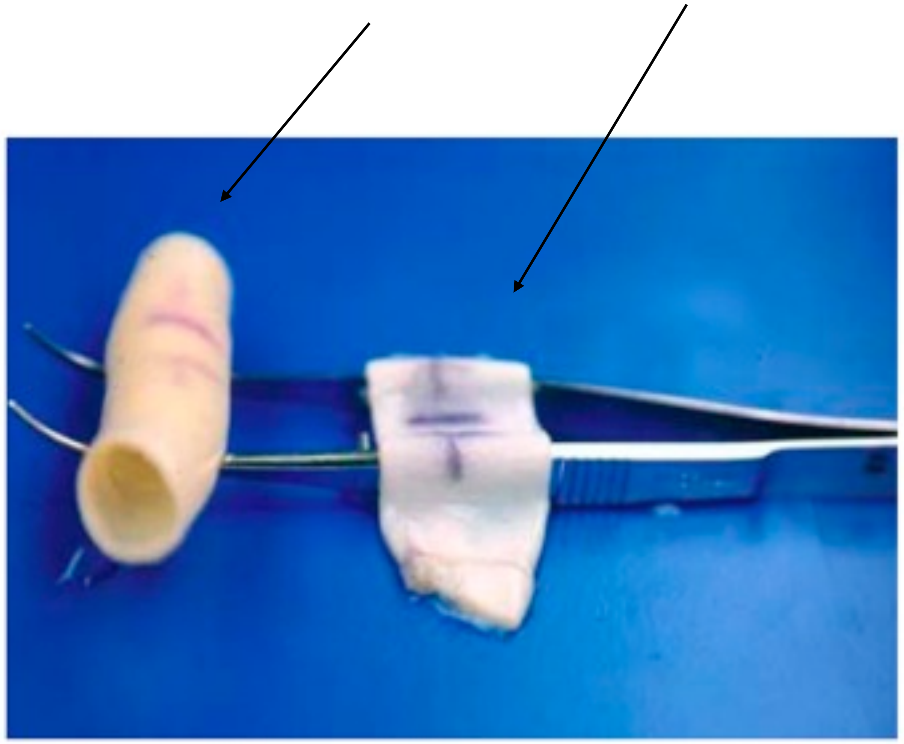
Picture from Holzapfel-Ogden

- Stimuli for growth
  - Increase in pressure → Increase in thickness (Thoma 1893)
    Normalize circumferential stress
  - Increase in blood flow rate → Increase in diameter (Thoma 1893)
    Normalize shear stress

Stress ⇒ Growth: arteries



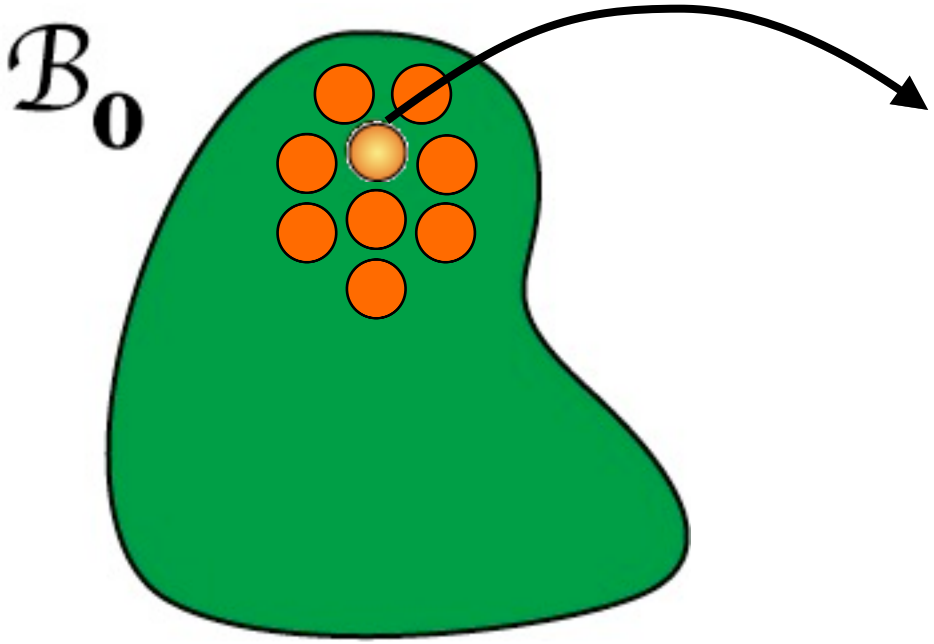
Media-intima and adventitia



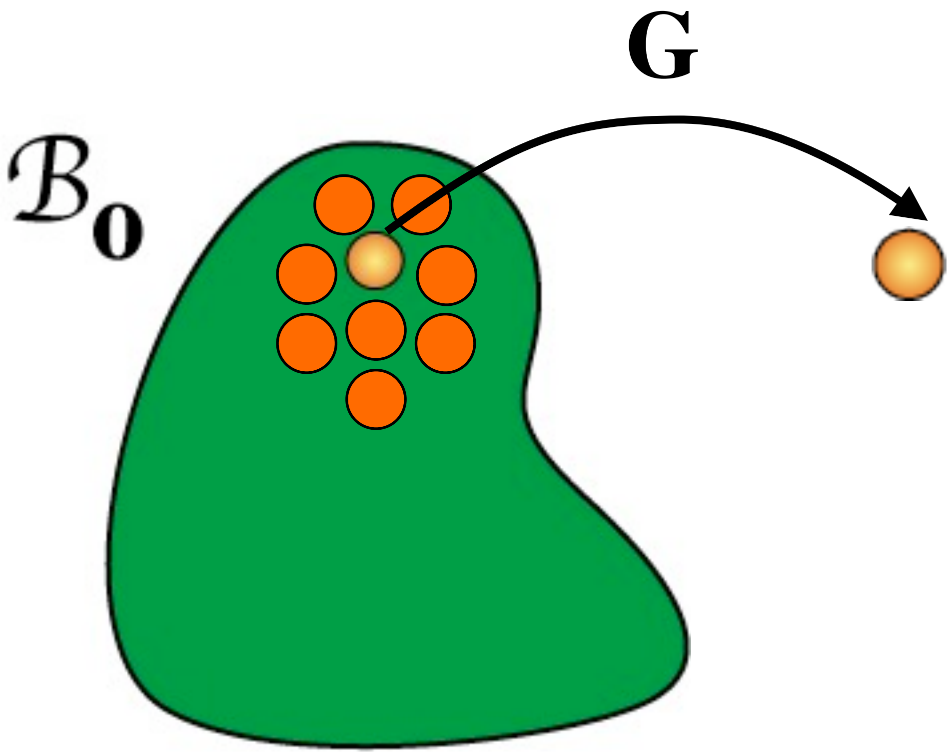
Picture from Holzapfel-Ogden

- Stimuli for growth
  - Increase in pressure → Increase in thickness (Thoma 1893)
    Normalize circumferential stress
  - Increase in blood flow rate → Increase in diameter (Thoma 1893)
    Normalize shear stress
  - Increase in axial load → Increase in length
    Normalize axial stress

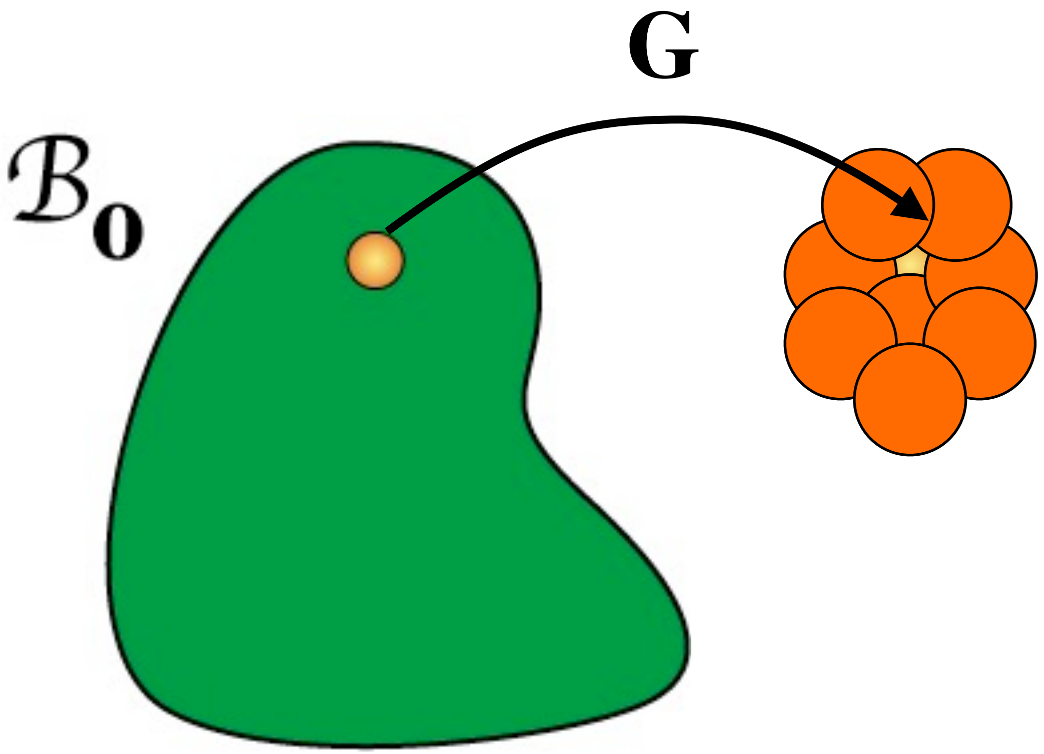
Growth ⇒ Stress



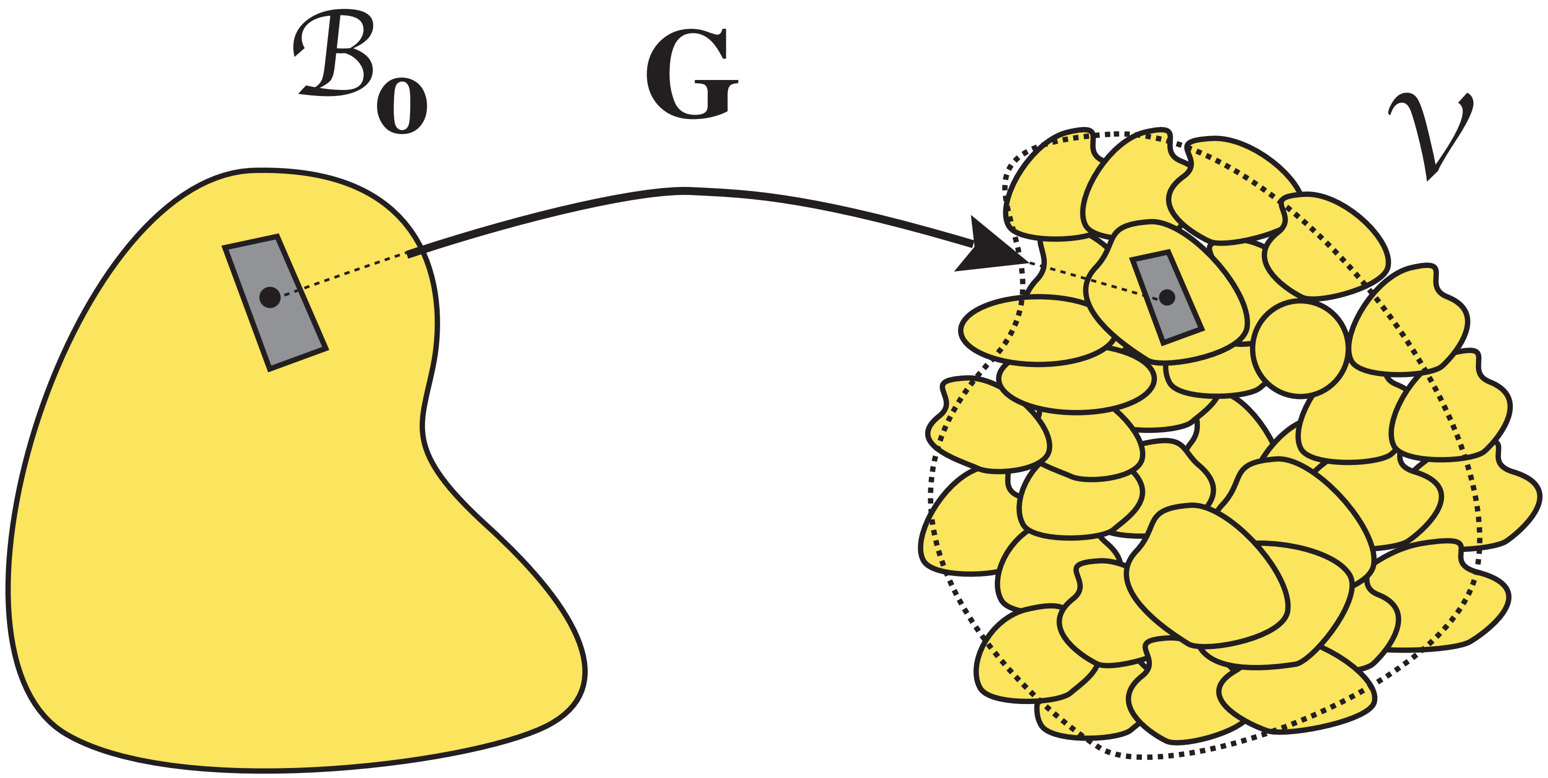
Growth ⇒ Stress



Growth ⇒ Stress



Growth ⇒ Stress

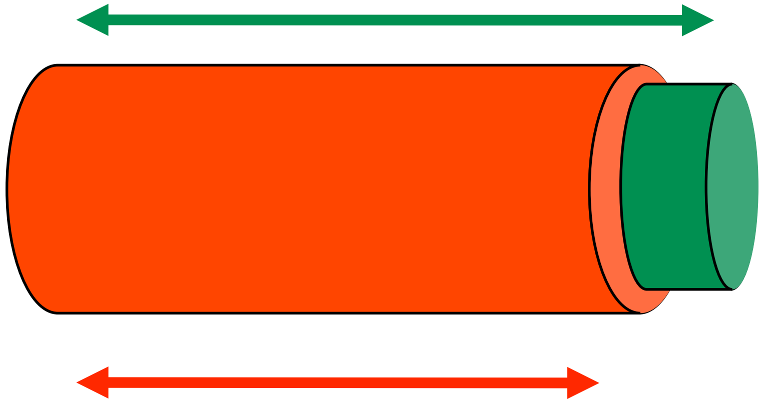


Intial reference configuration

Virtual stress-free configuration

Growth ⇒ Stress: residual stress

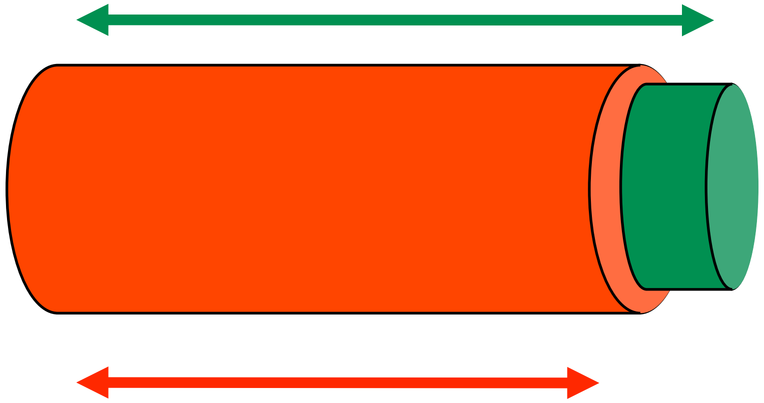
- Differential growth



Green grows faster than red.
So red is in tension, green is in compression
This stress is called residual stress.

Growth ⇒ Stress: residual stress

- Differential growth



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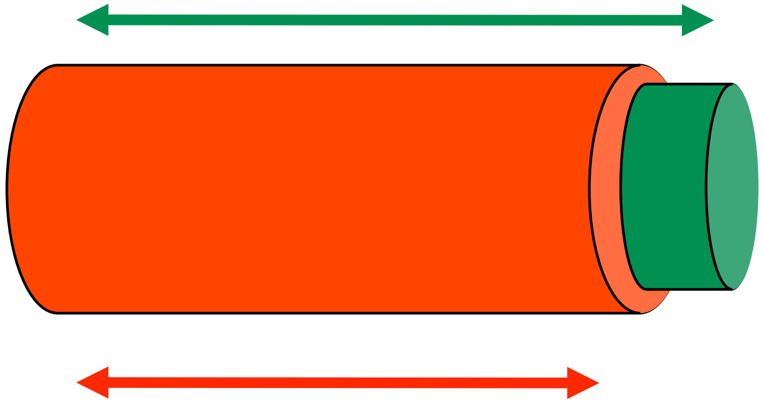
Growth ⇒ Stress: residual stress

- Differential growth



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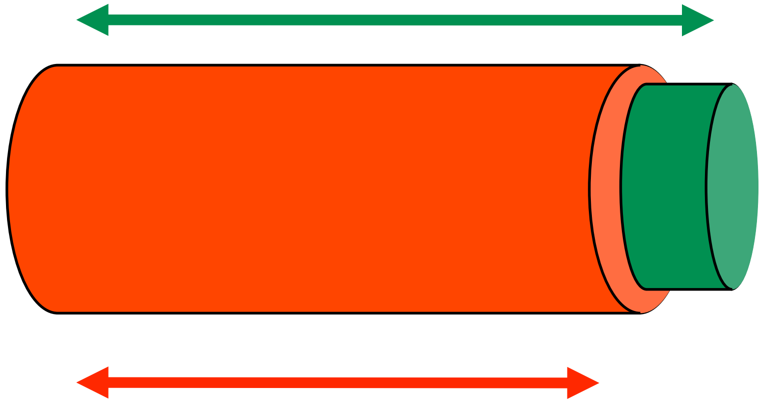

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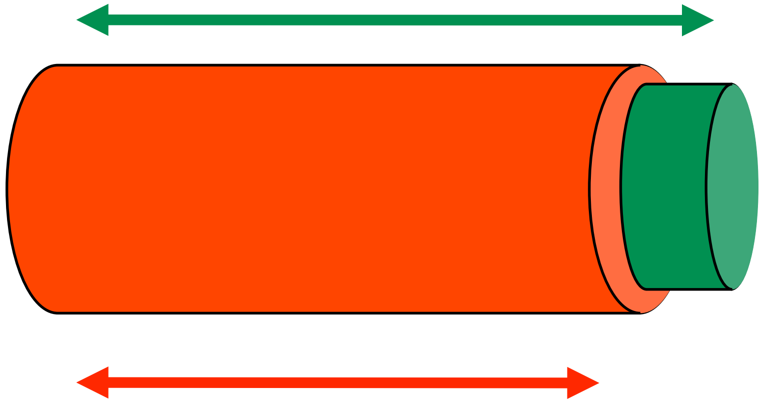
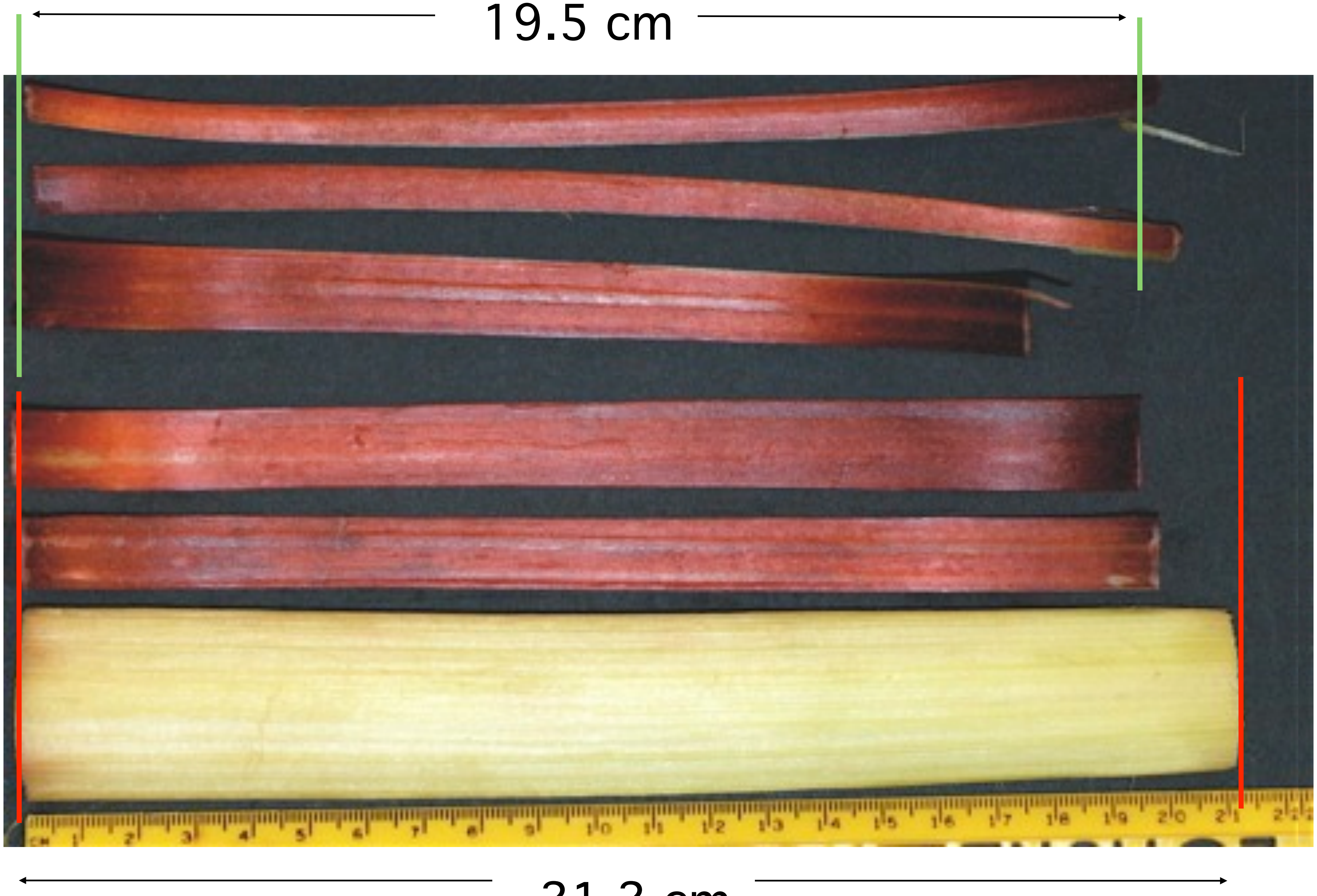


Green grows faster than red.
So red is in tension, green is in compression
This stress is called residual stress.

Rhubarb: *Rheum rhabarbarum*
Tissue tension Brucke 1848

Growth ⇒ Stress: Rhubarb



19.5 cm

- 2-3%

21.2 cm

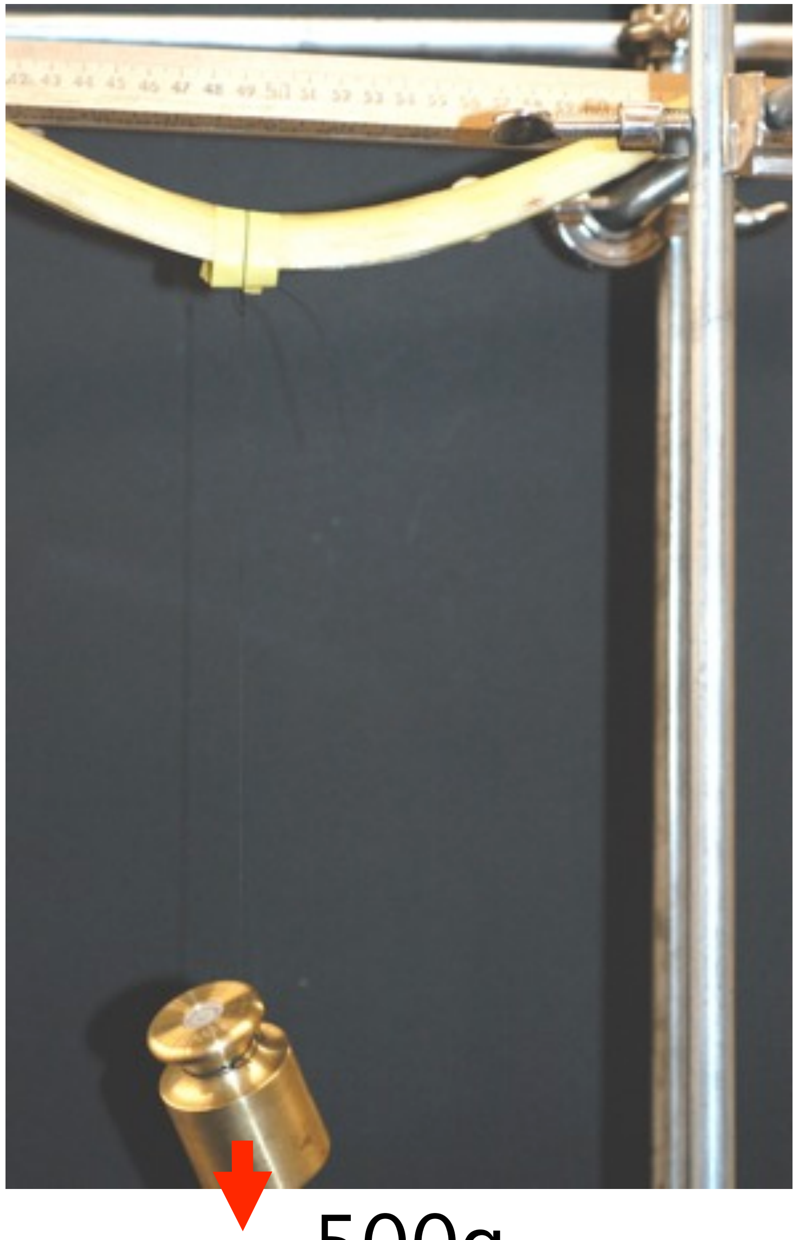
+ 6 %

Growth ⇒ Stress: Rhubarb

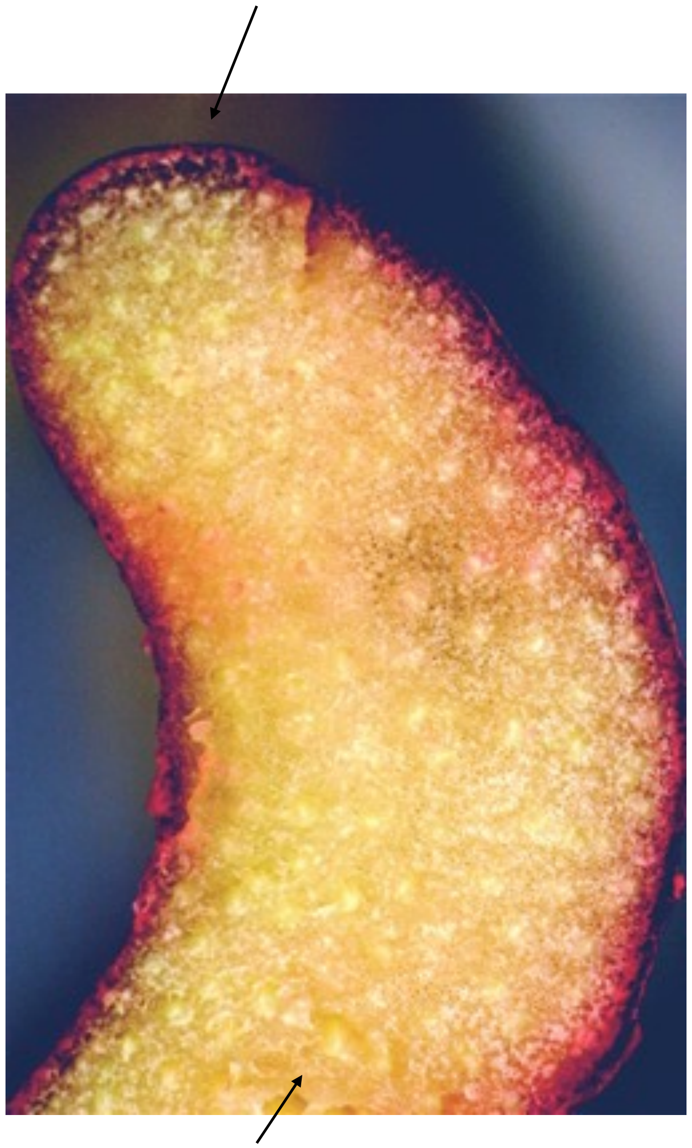
Before

2kg

After

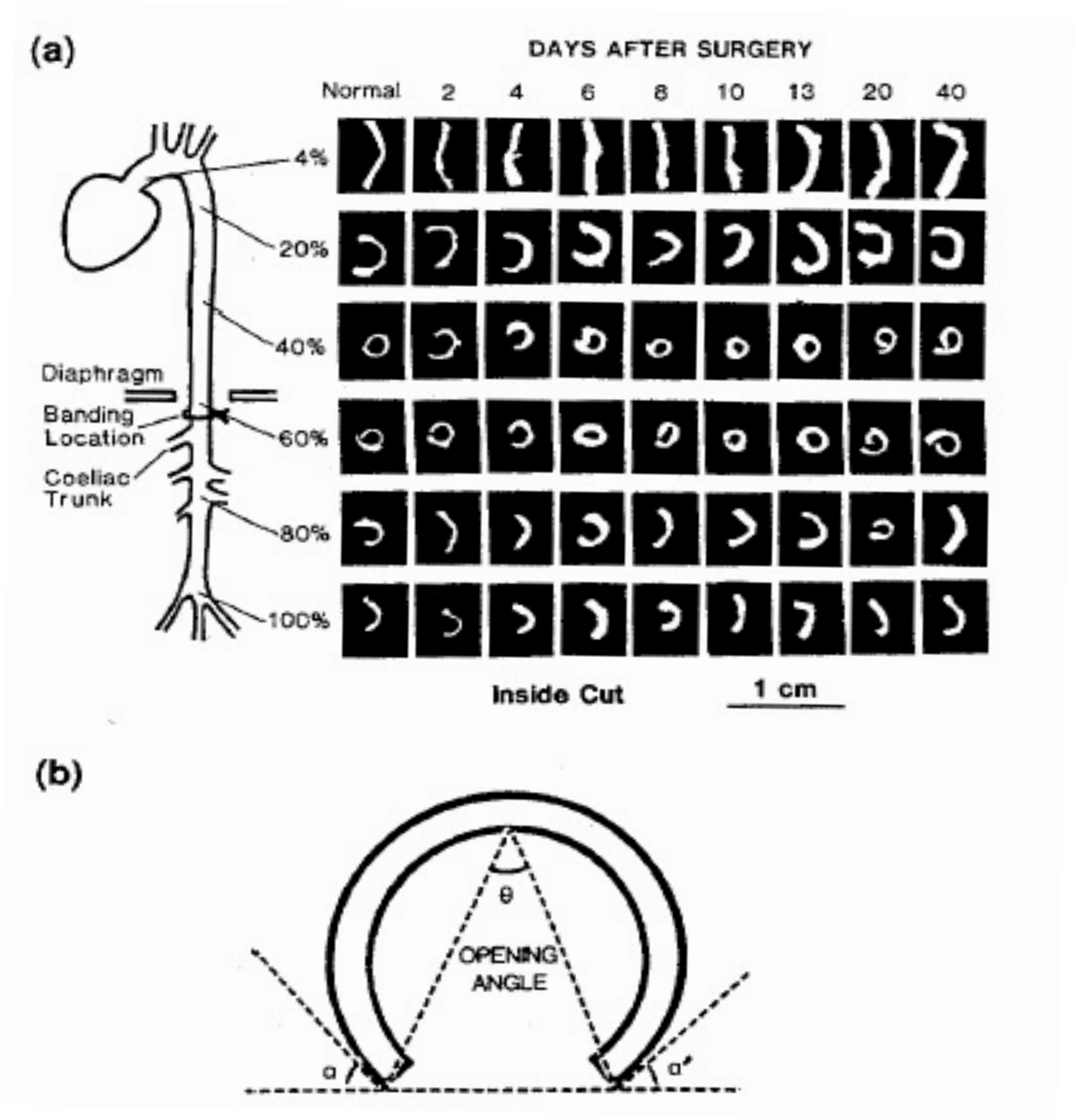


500g



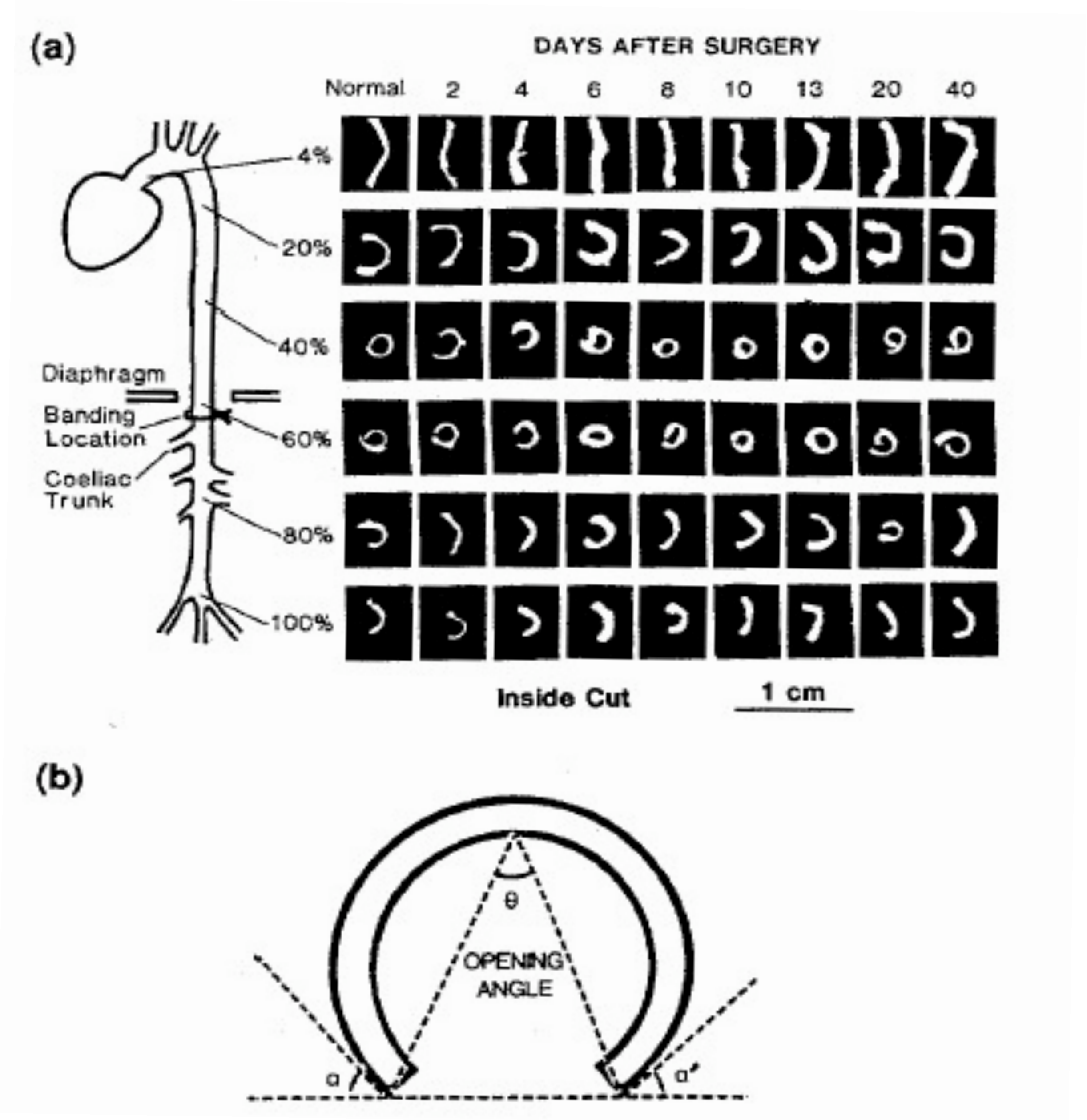
“We have here the case of an elastic stiff body consisting of two parts, each in a high degree flexible and by no means stiff; only in their natural connection do the epidermal tissue and internal tissues together form an elastic rigid body.” *(Sachs, 1875)*

Growth ⇒ Stress: arteries



Fung (the cover of his 1993 Biomechanics book)

Growth ⇒ Stress: arteries



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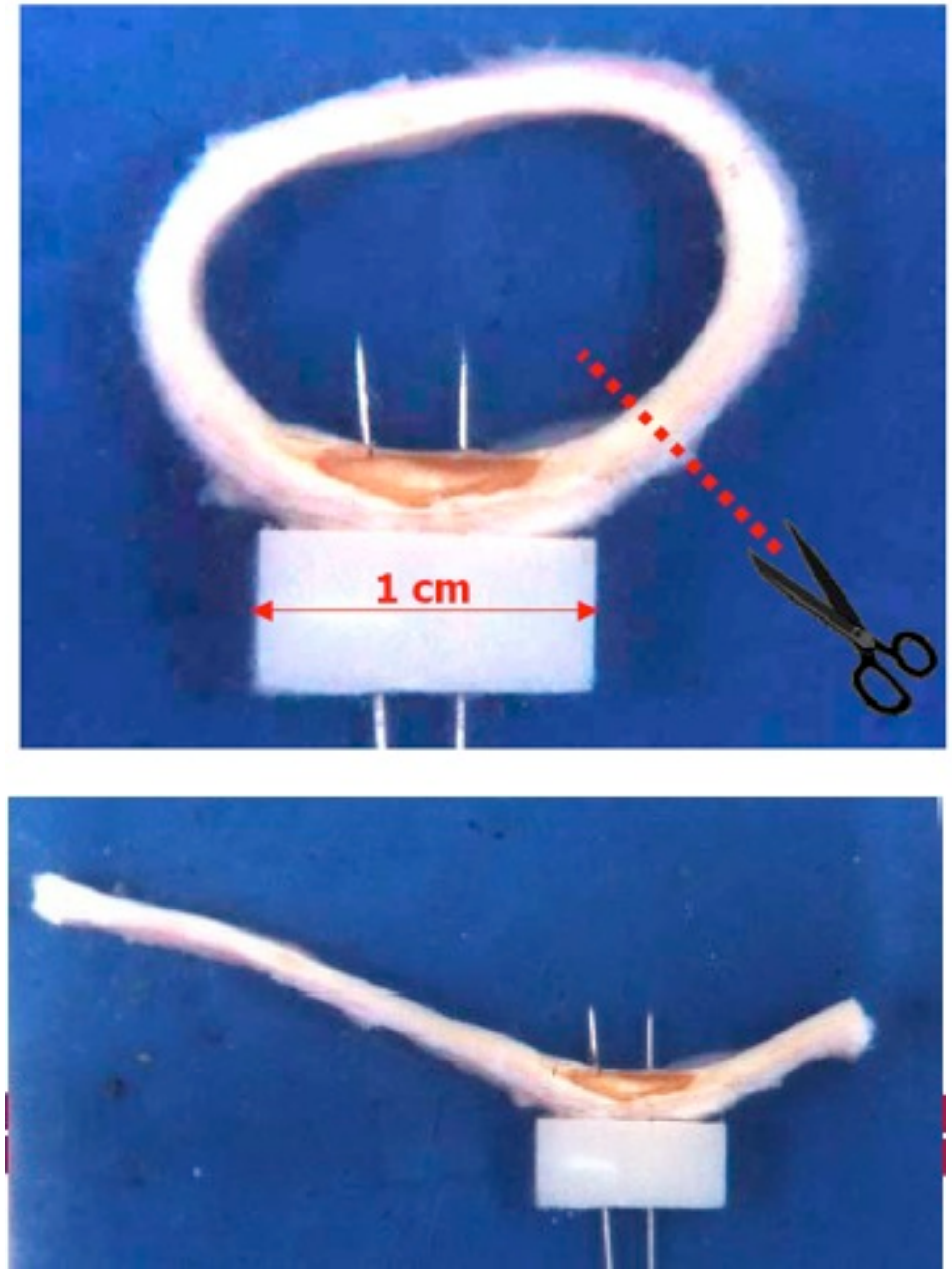
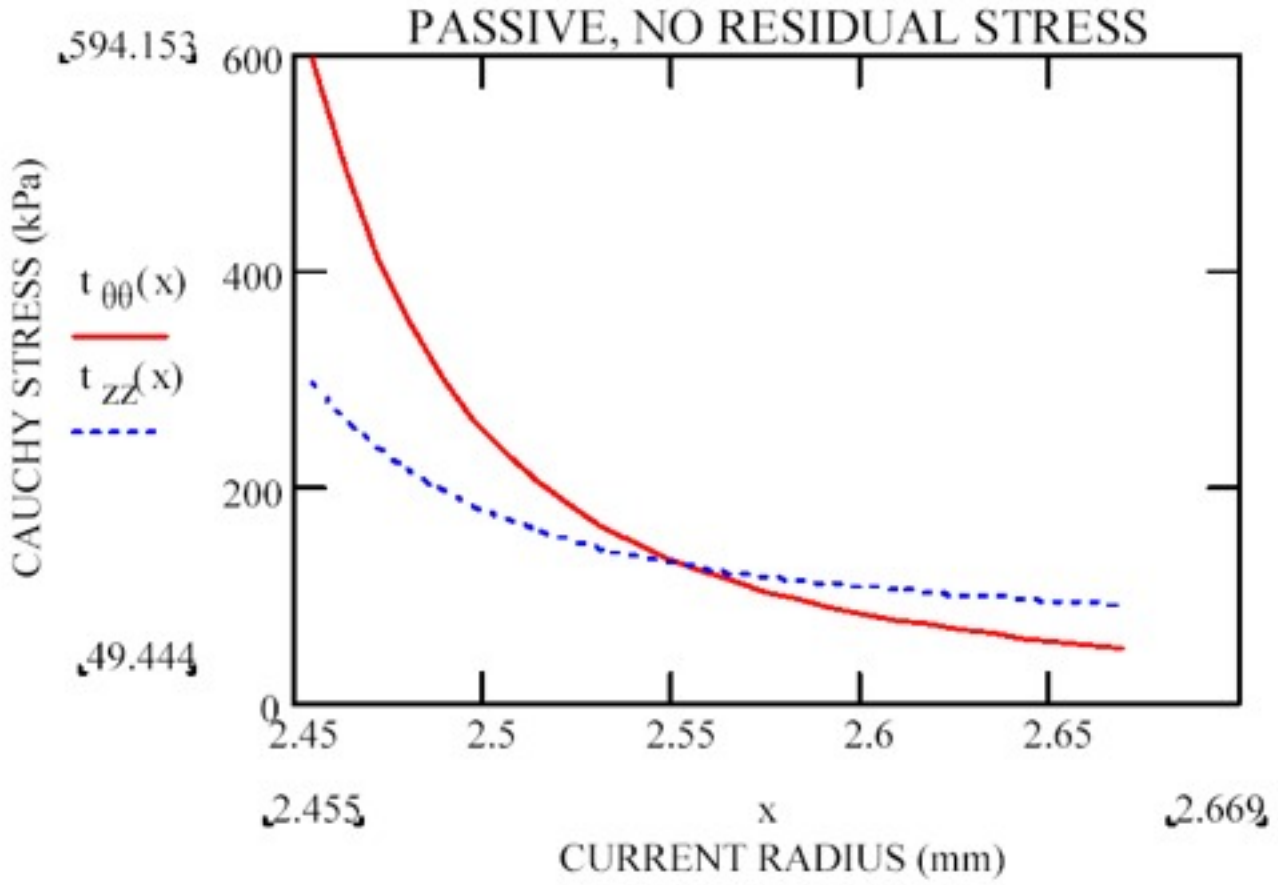
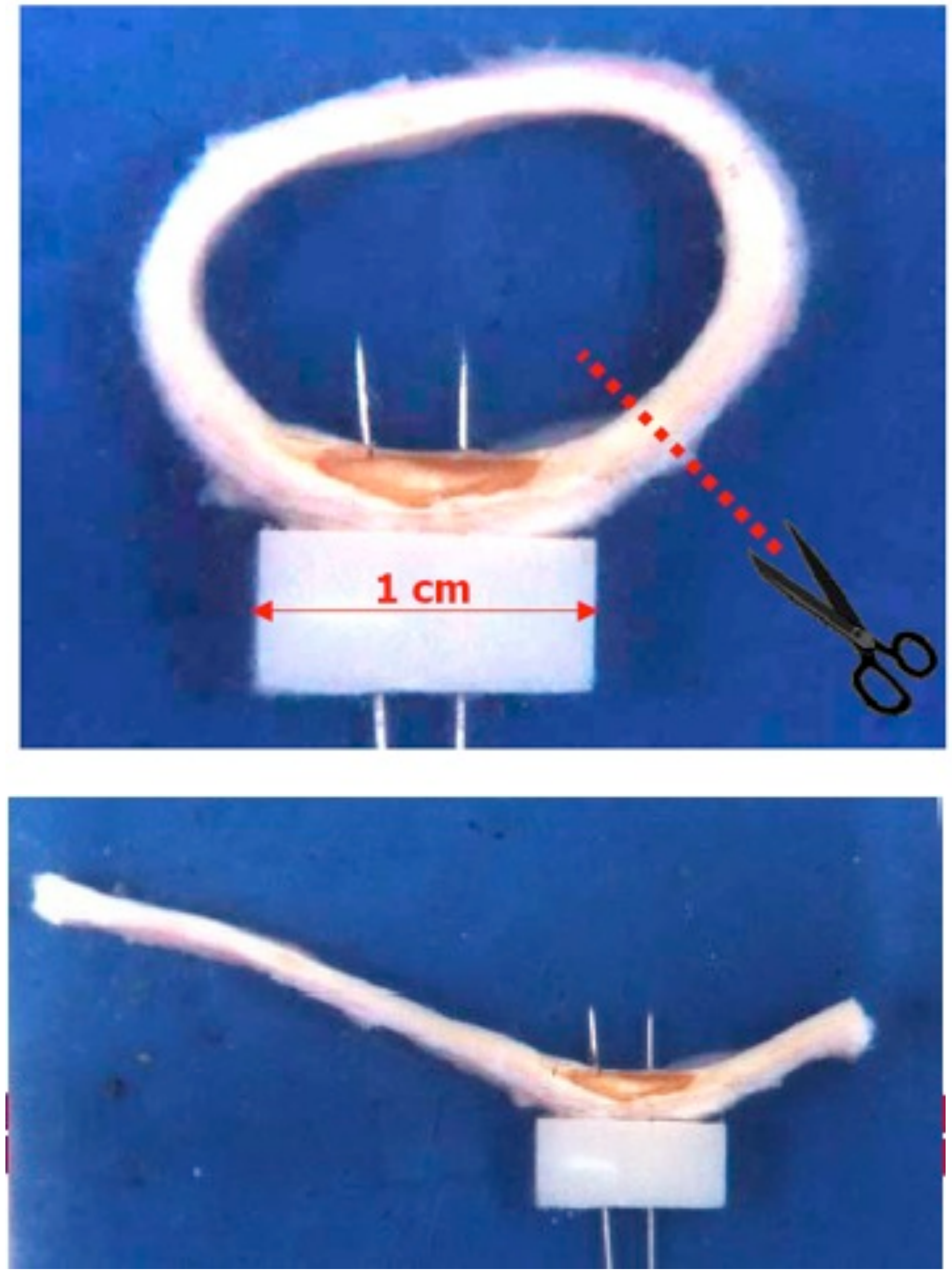


Picture from Holzapfel-Ogden

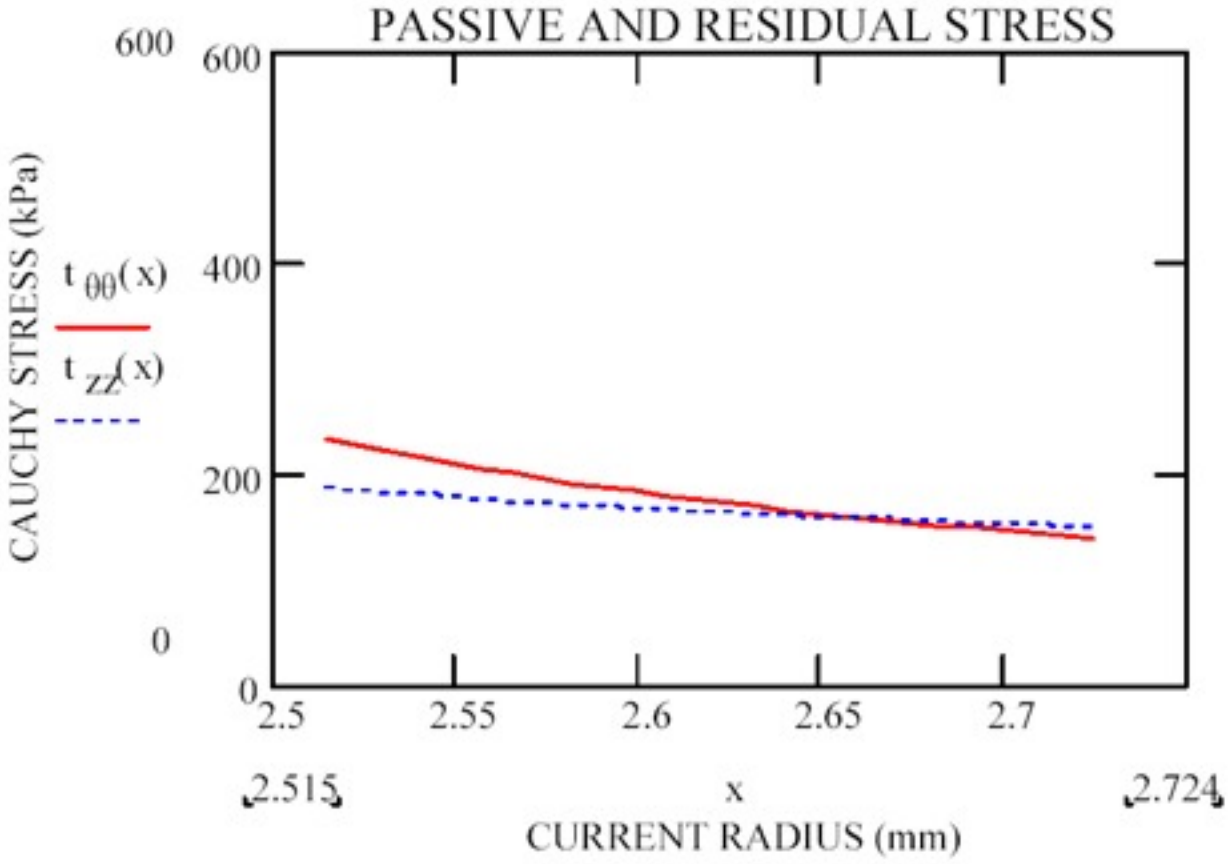
Growth ⇒ Stress: arteries



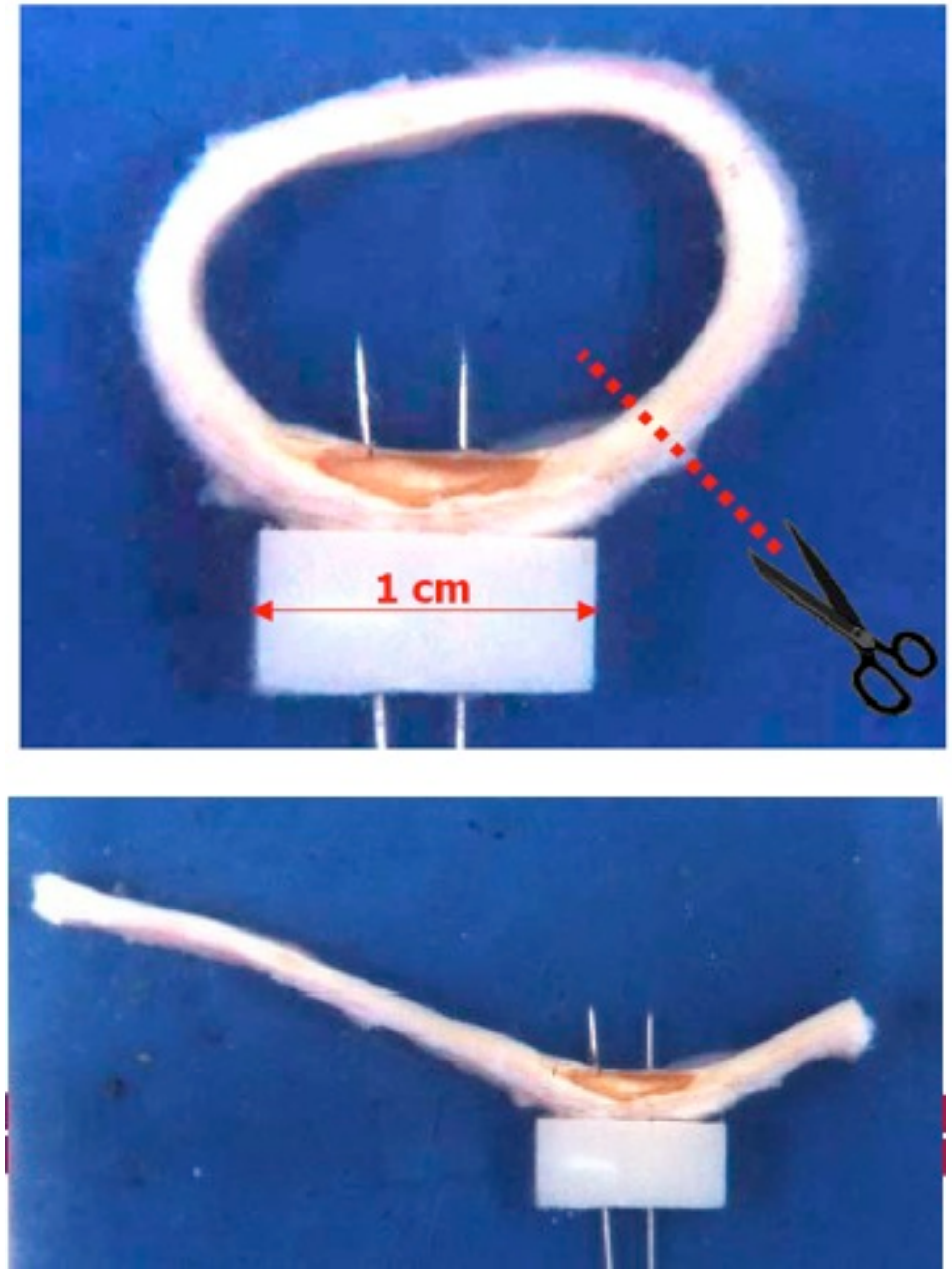


Picture from Holzapfel-Ogden

Growth ⇒ Stress: arteries



Ref: Humphrey ('03)

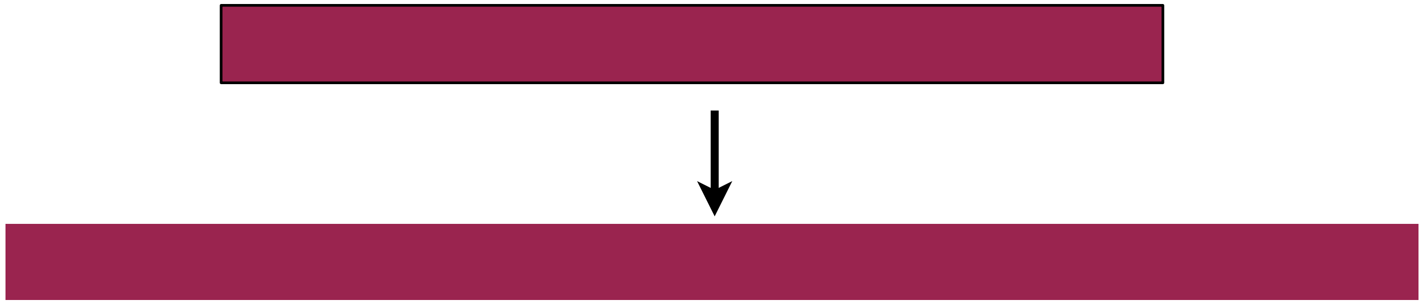


Picture from Holzapfel-Ogden

Growth ⇒ Stress ⇒ Growth

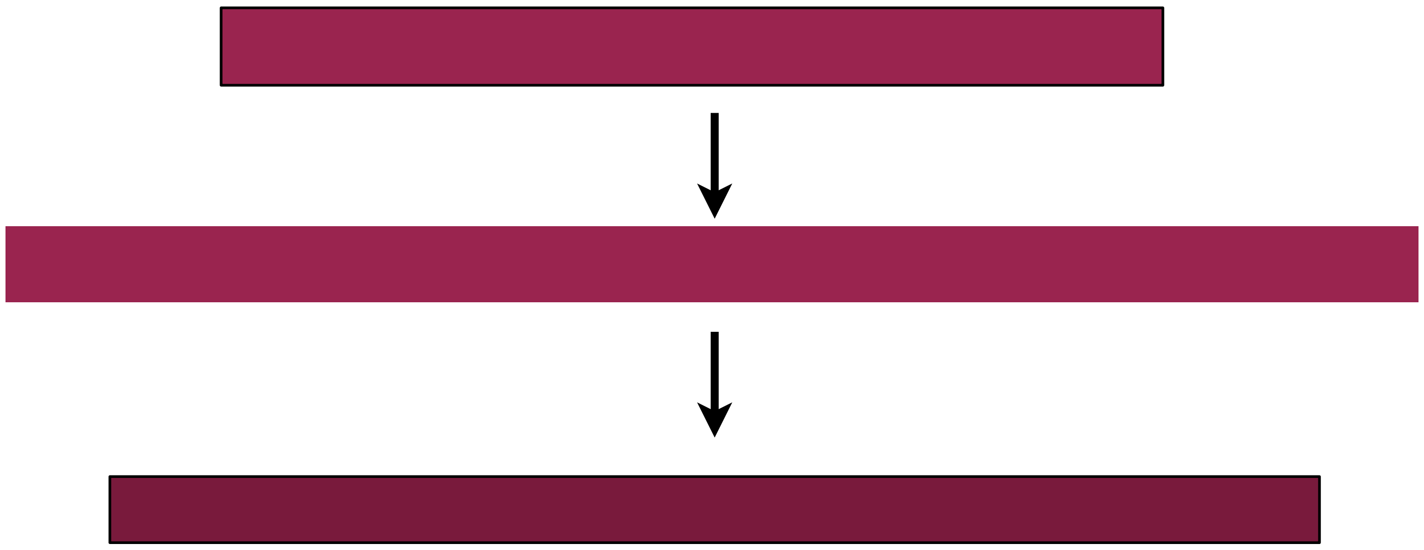
L_0 Unstressed

Growth ⇒ Stress ⇒ Growth



L_0	Unstressed
$1.6\ L_0$	Physiological

Growth ⇒ Stress ⇒ Growth



L_0 Unstressed

$1.6\ L_0$ Physiological

$1.3\ L_0$ Experiment

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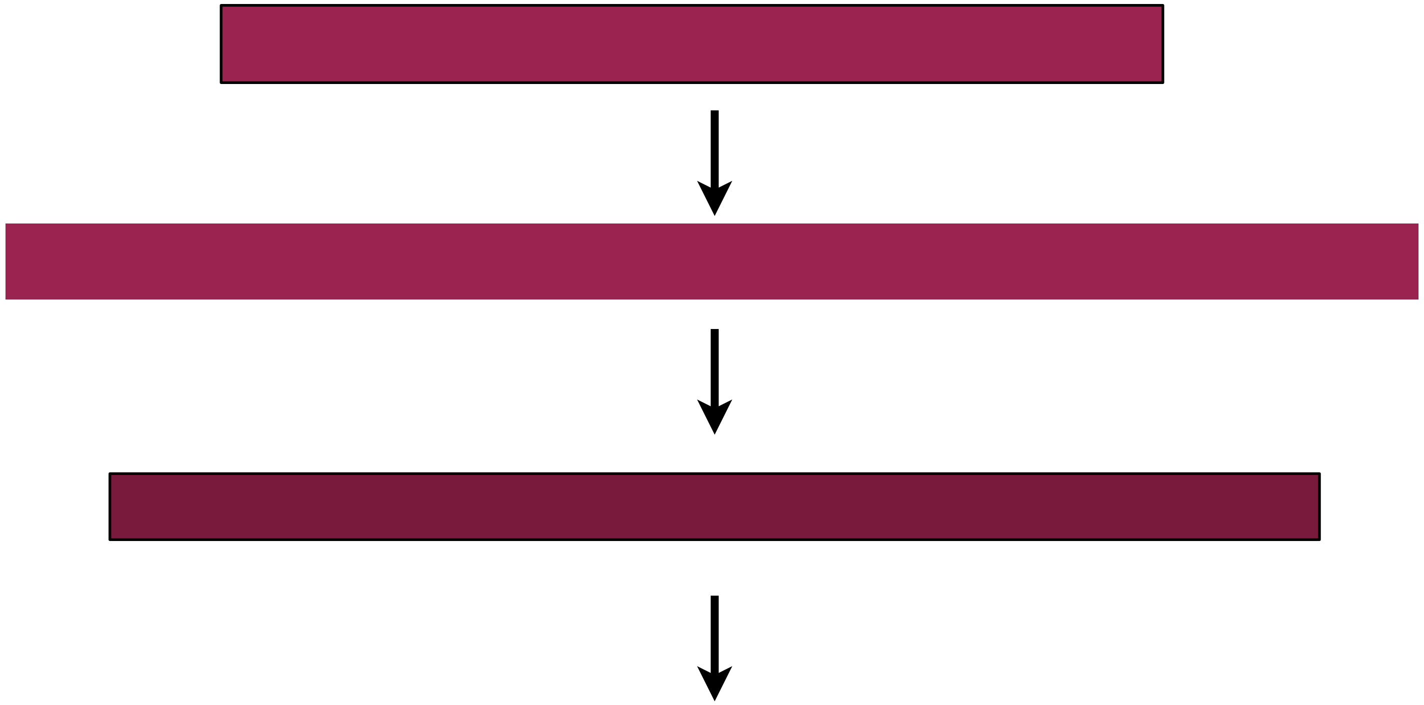

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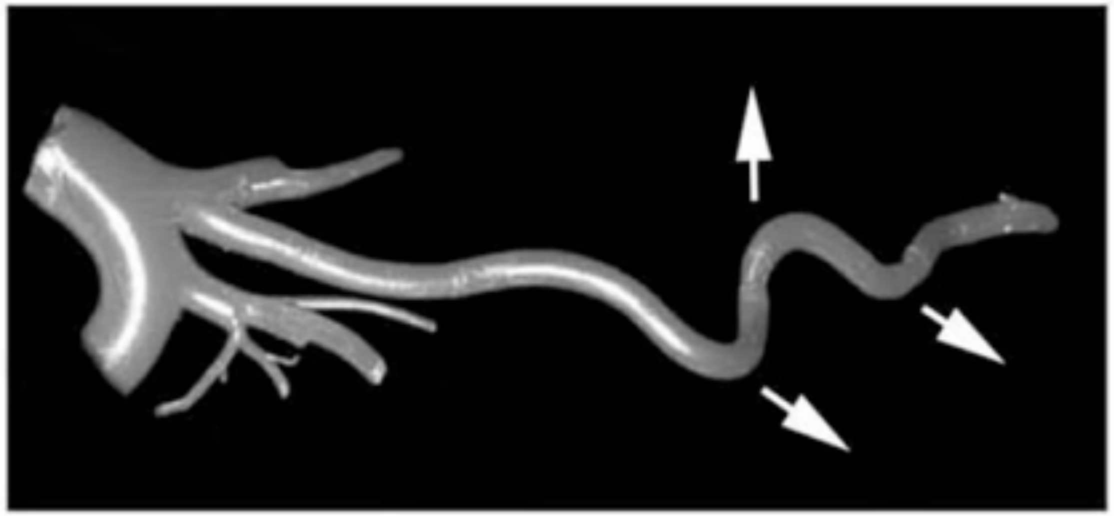

Jackson et al. 2005

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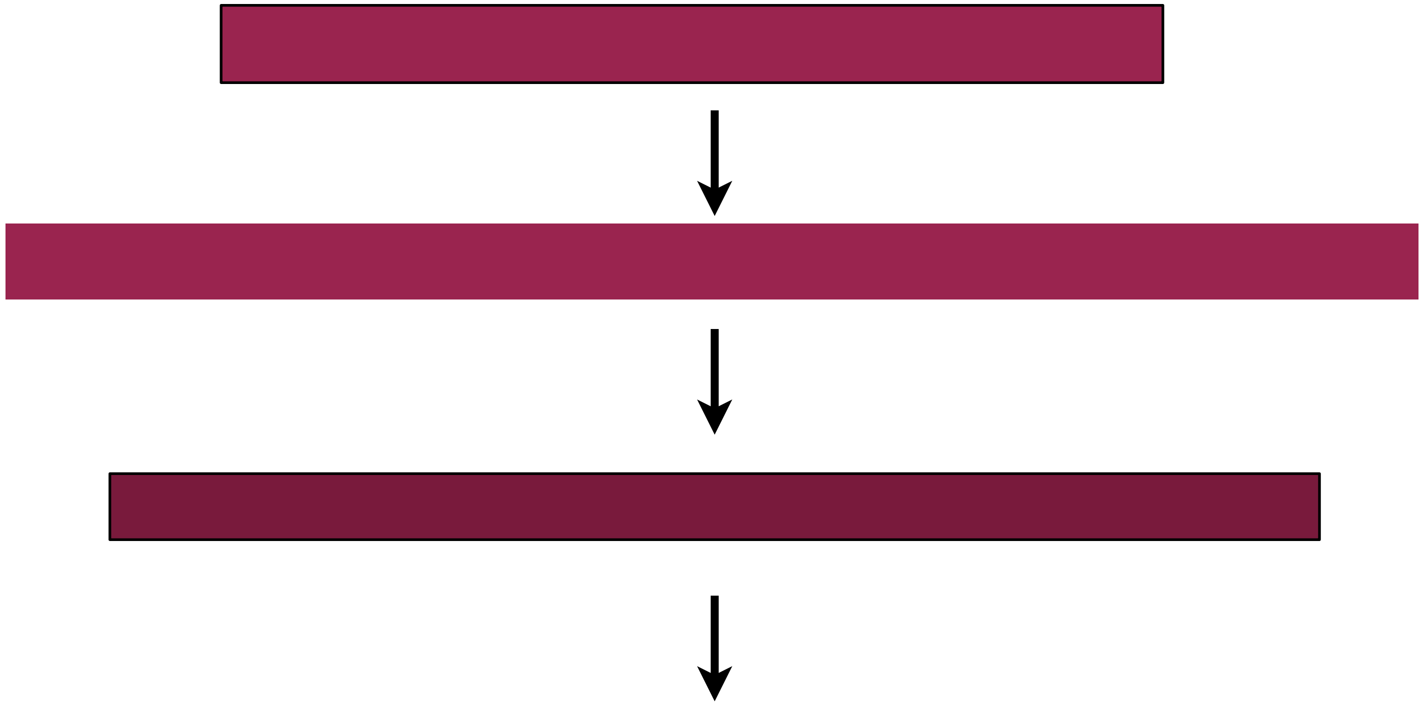

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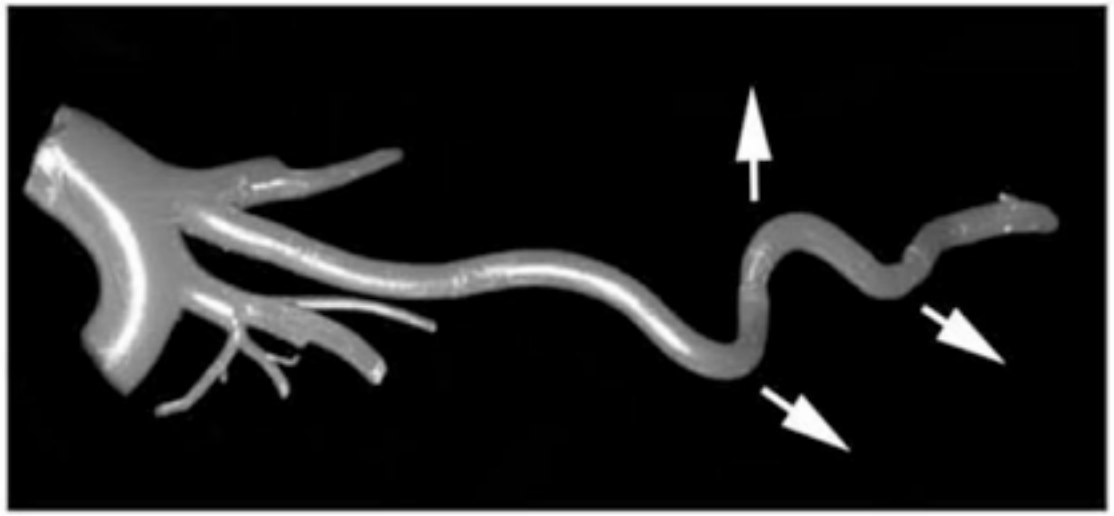

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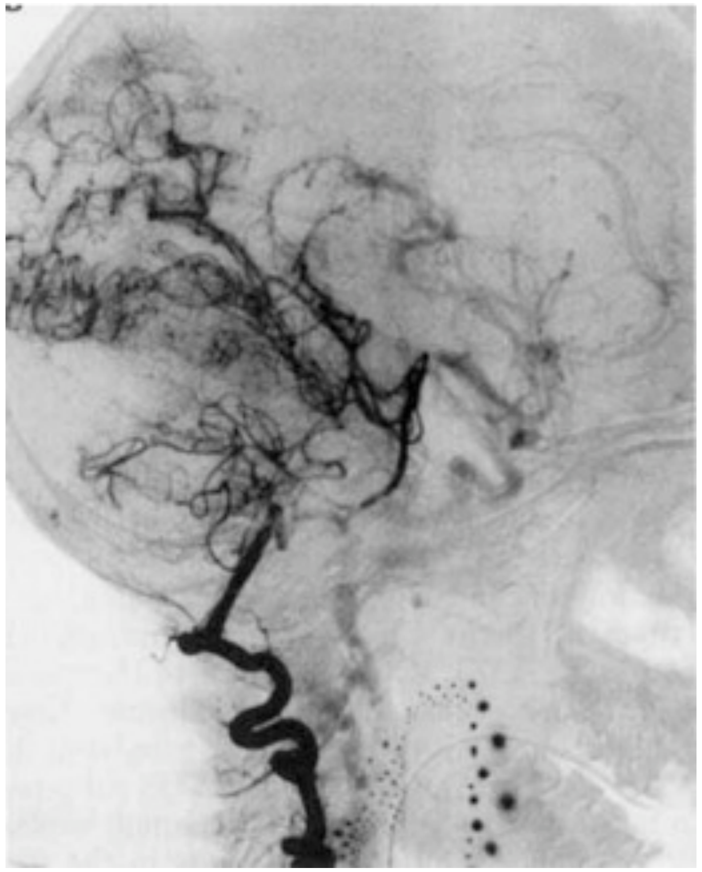

Adès et al. 1996

Growth in the kitchen

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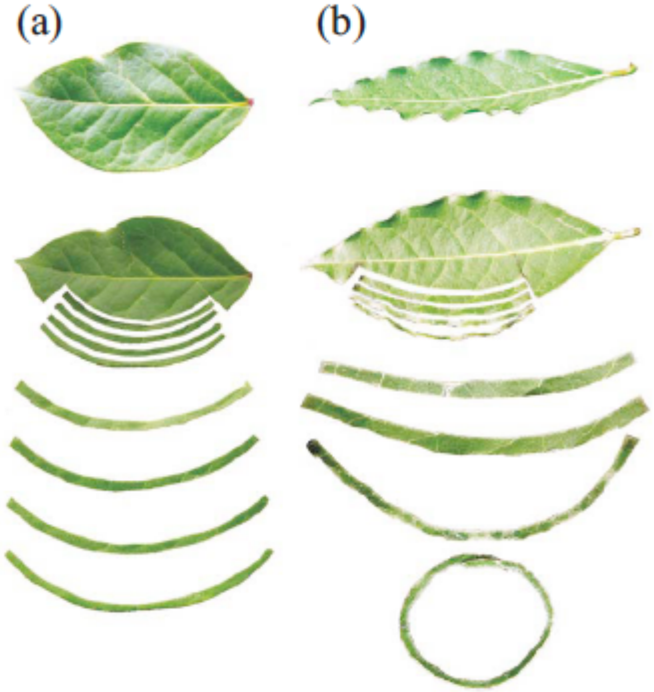
BACON

IS MEAT CANDY

Growth in the kitchen



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See work by Marder-Roman-Swinney,
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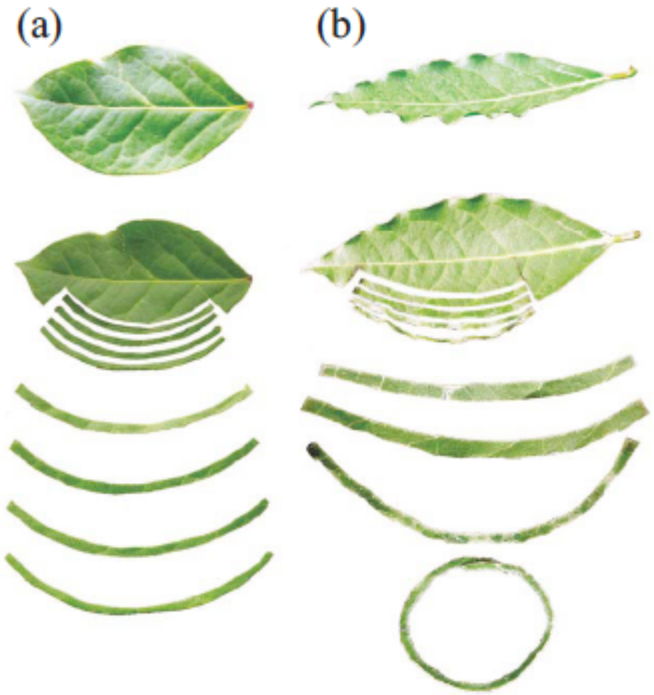

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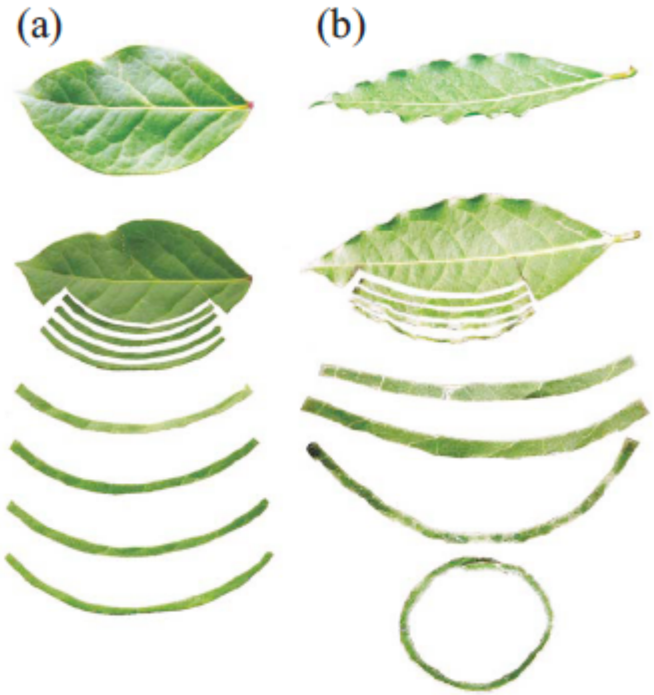
Stem pithiness



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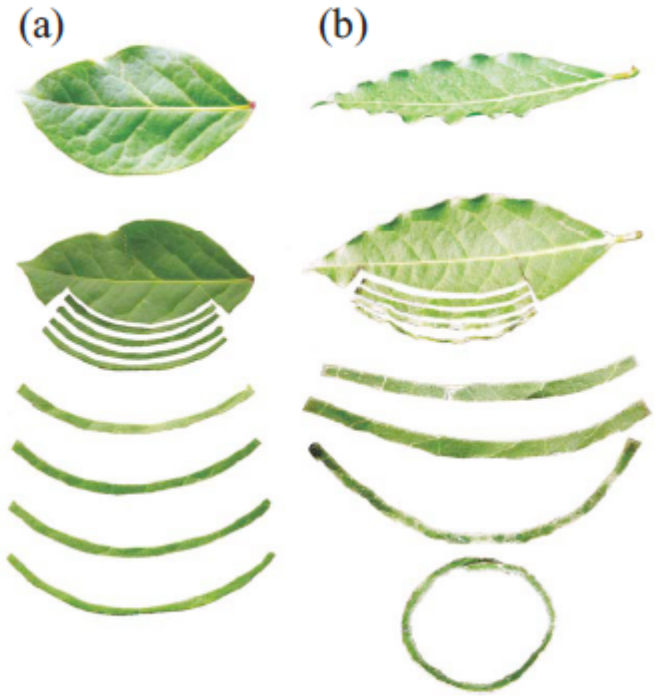

See work by Marder-Roman-Swinney,
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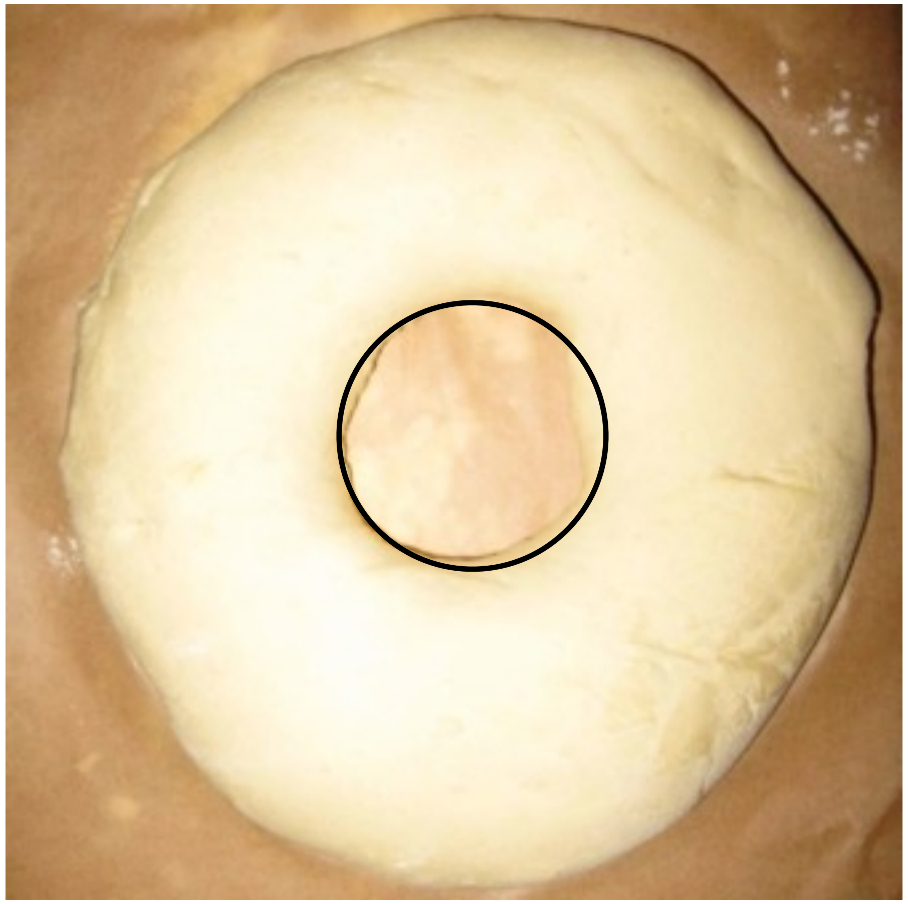
> *"I will refer only to the one obvious fact that while the pith is no longer able to grow in proportion, it becomes ruptured while a cavity arises in the interior. This may be easily observed in the flower stems of the Teazel and Dandelion,…"*

Sachs 1887

Growth in the kitchen



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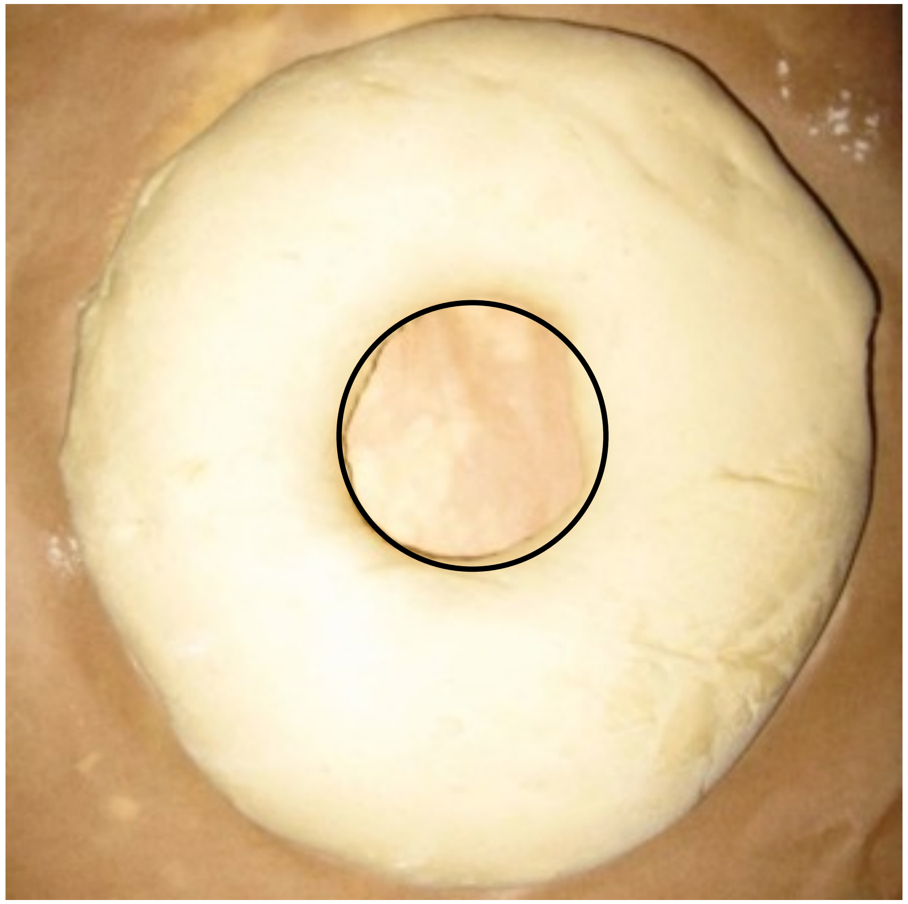

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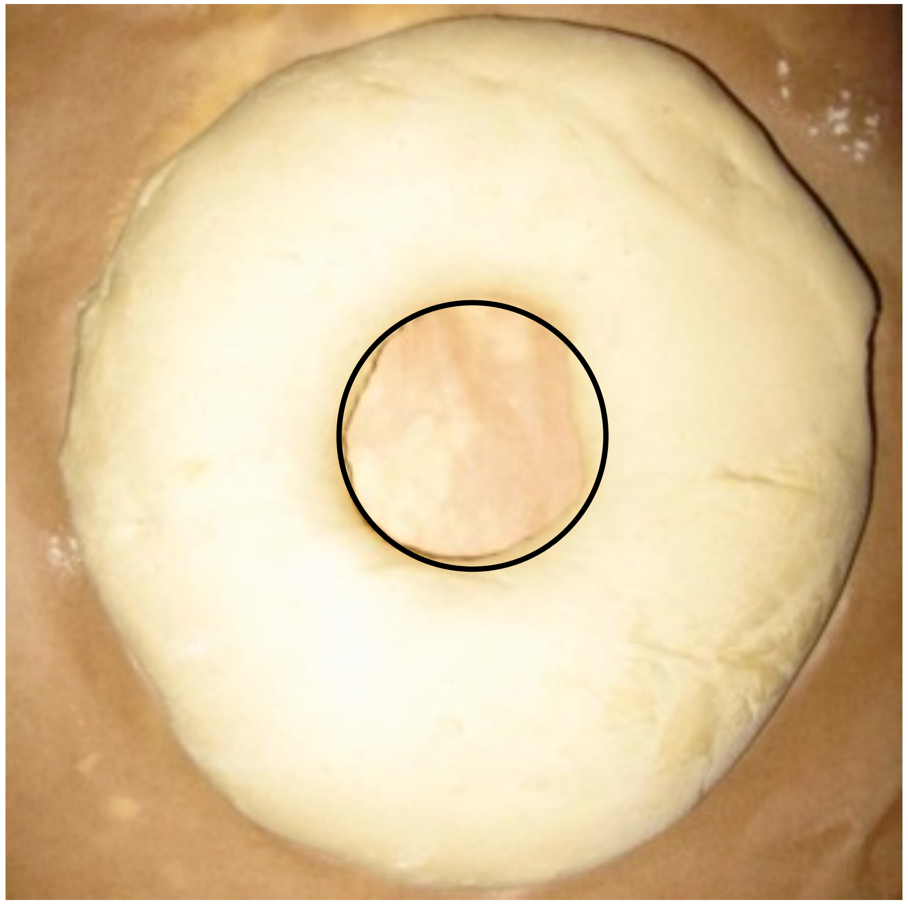

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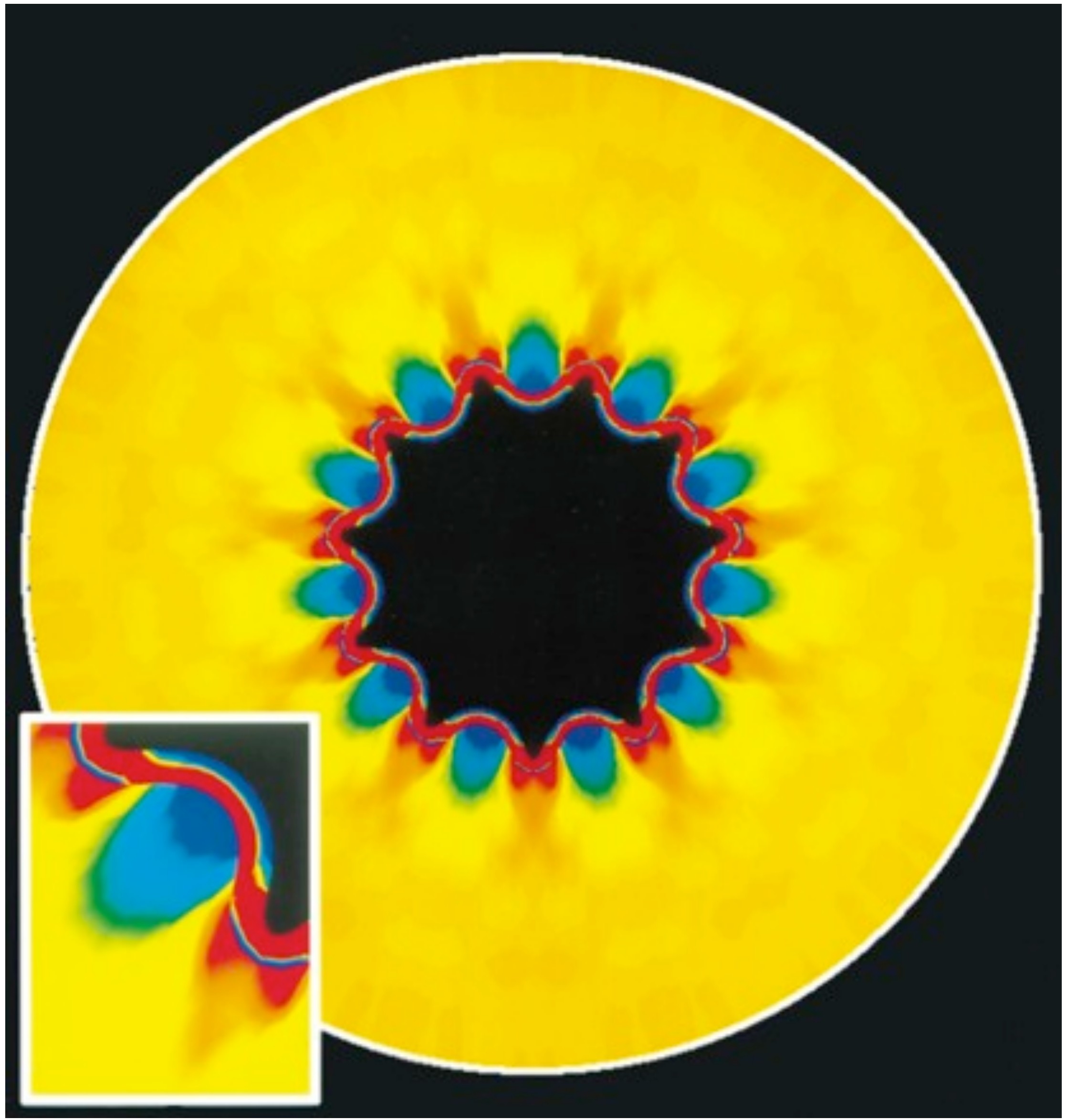
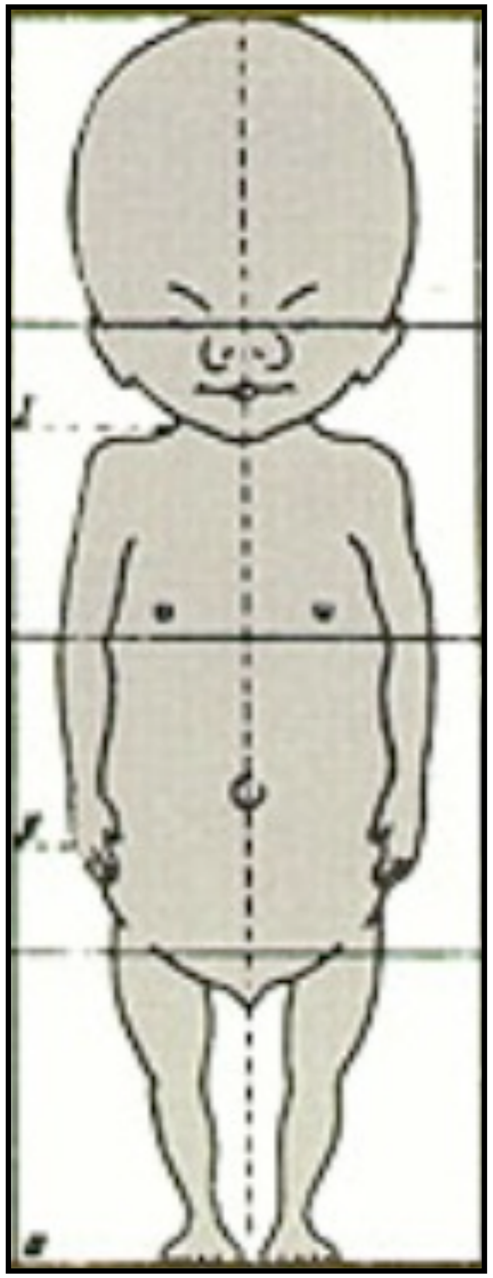


Mucosal folding in airways
Wiggs et al. ’97 (J. Appl. Physiol.)

Part 2. Modeling growth

- 2. Modeling
  - Growth on the line
  - Kinematics of growth
  - Incompatibility
  - Mechanics
  - Elasticity
  - Stability



General framework

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- Need for nonlinear elasticity

  Large deformations, material non-homogeneous, anisotropic, with nonlinear response and (often) residual stress, arising from growth

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  Multiple length scales:

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  - Tissues
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Forcing	1-10^{3}s
Growth	10^{5}-10^{7}s

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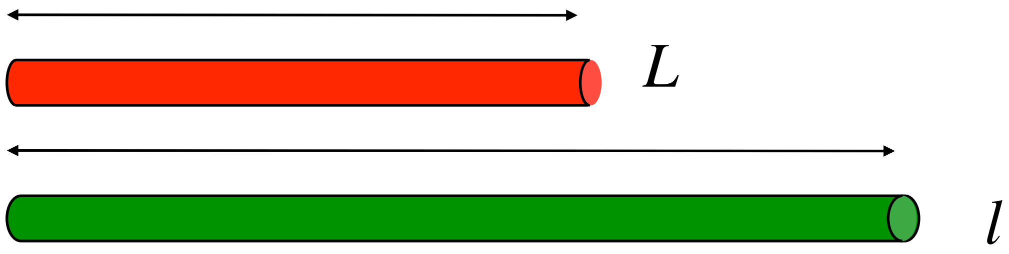
Remodeling

Evolution of material properties (density, moduli, viscous response,...)

Modeling growth on the line

$$\boxed{\left|\frac{\partial s}{\partial S}\right| = \lambda}$$



Modeling growth on the line

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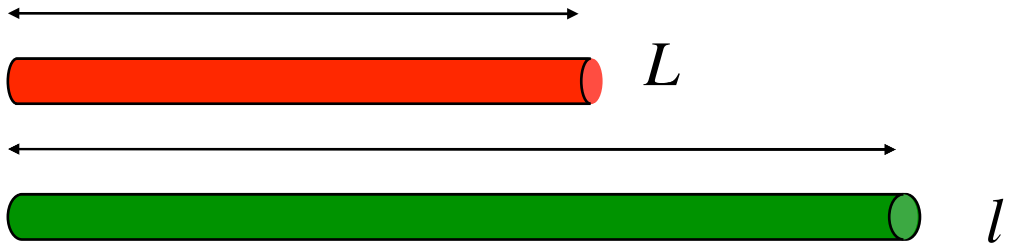

e.g. homogeneous $\lambda = l/L$

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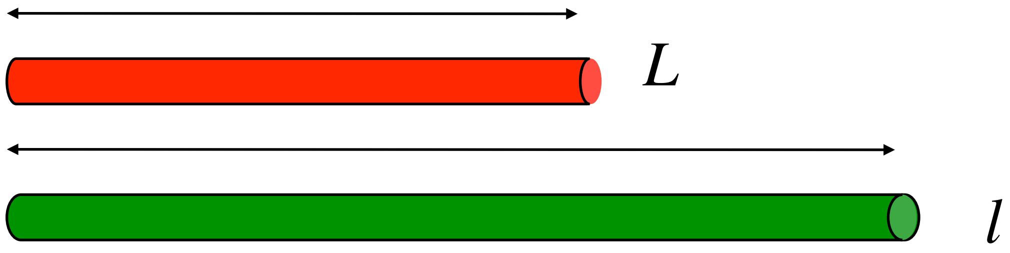

e.g. homogeneous $\lambda = l/L$

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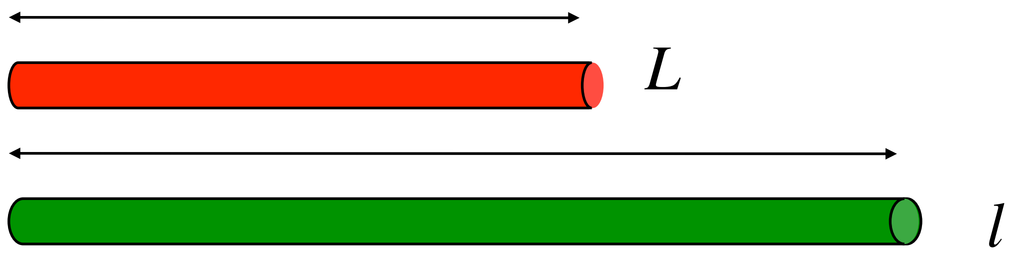

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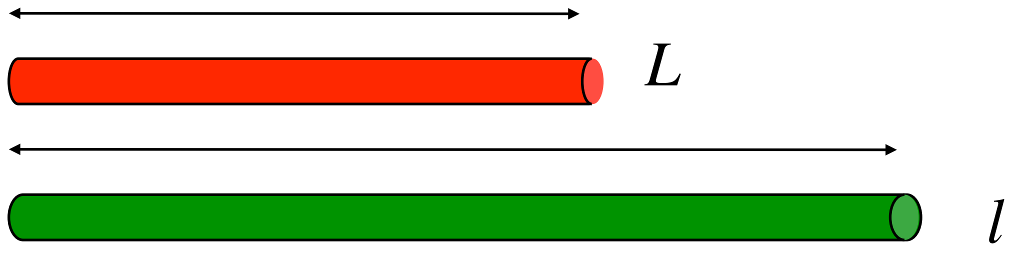

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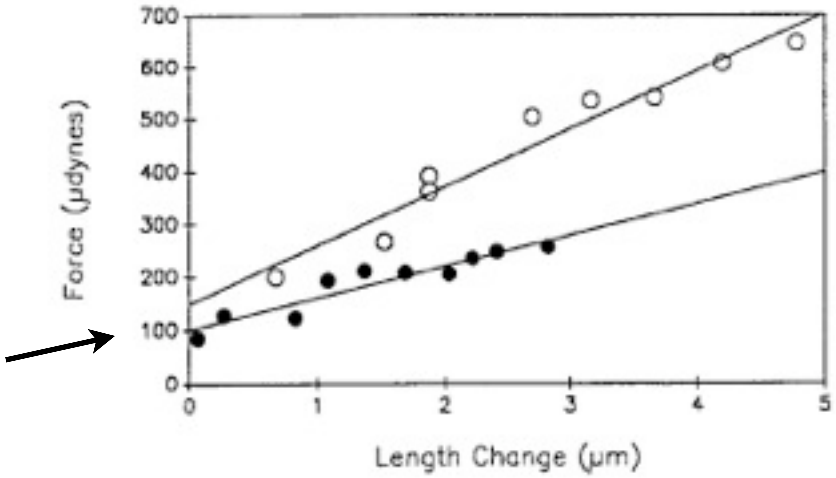

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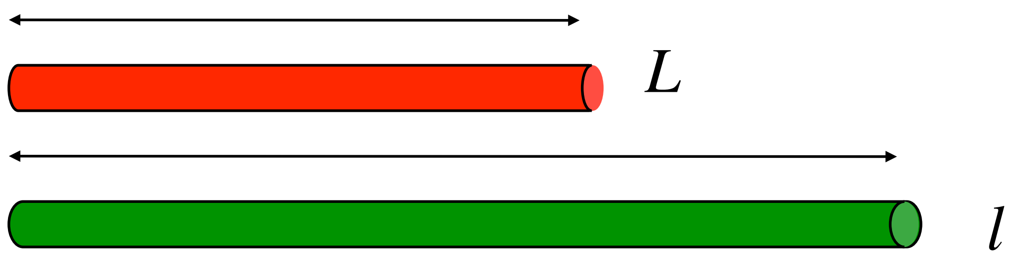

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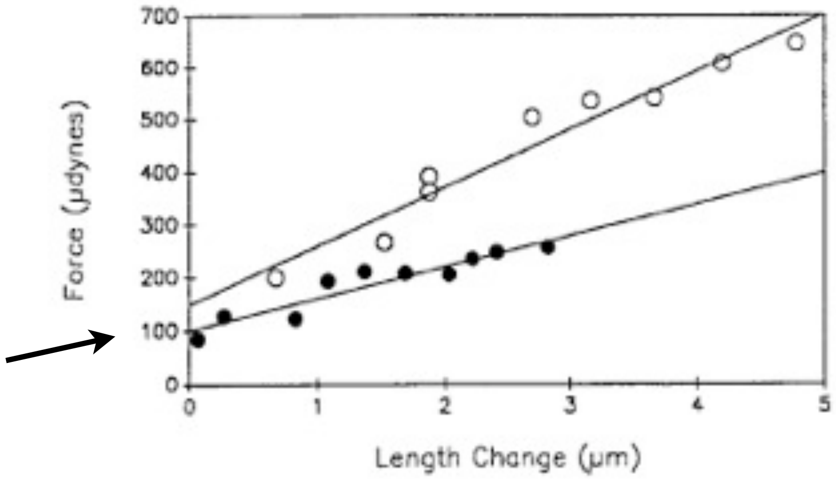

- 4. Growth law

$$\boxed{\frac{\partial\gamma}{\partial t} = G(s, S, t; \gamma, \sigma)}$$

Modeling growth on the line

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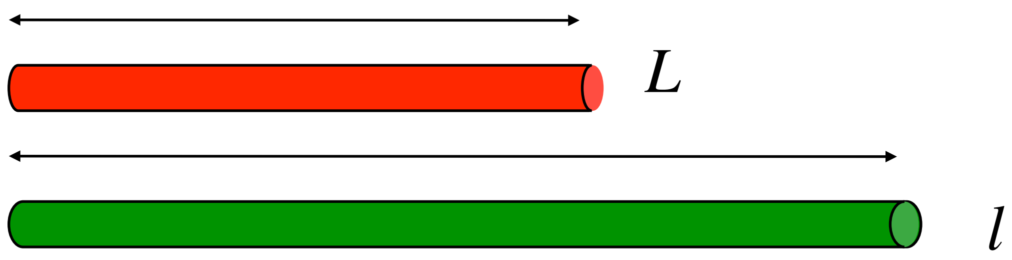

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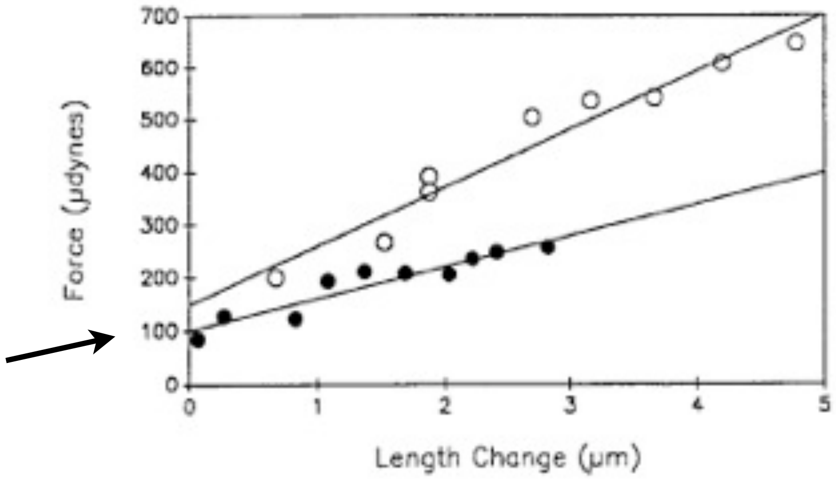

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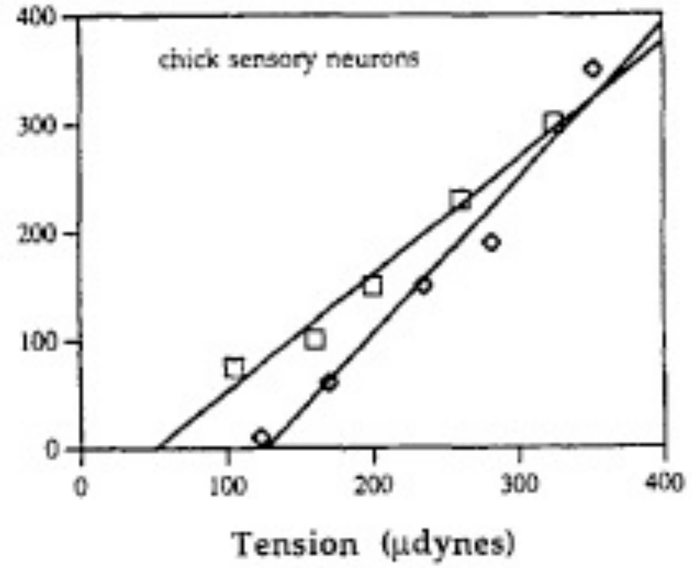
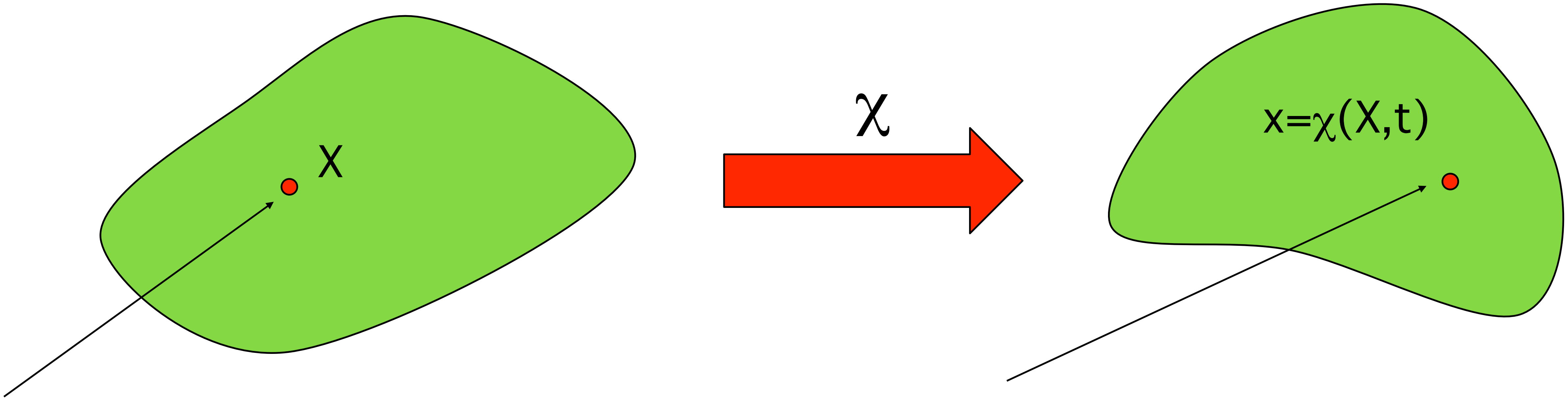


1. Kinematics of growth

Let **X** be a *reference* configuration of an elastic body and **x** the *current* configuration after growth.



Let $\mathbf{F}=\mathbf{Grad}(\chi)$ be the geometric deformation tensor.

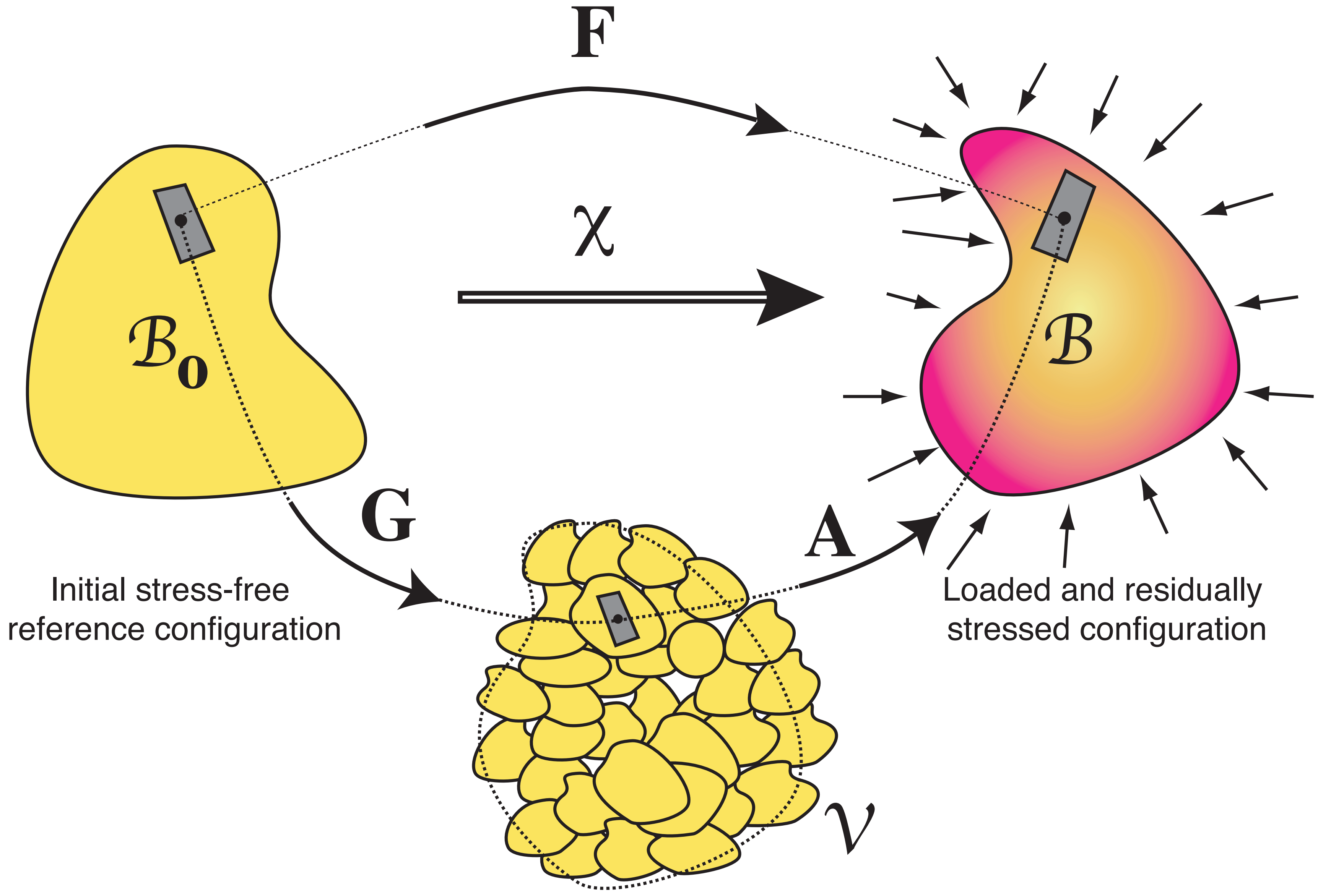
Fundamental assumption of **morpho-elasticity**

$$\boxed{\mathbf{F}=\mathbf{AG}}$$

where **G** and **A** are, the GROWTH and ELASTIC deformation tensors.

1. Kinematics

Decomposition: $\mathbf{F}=\mathbf{AG}$ $\mathbf{G}$: Growth deformation $\mathbf{A}$: elastic deformation.



Initial stress-free reference configuration

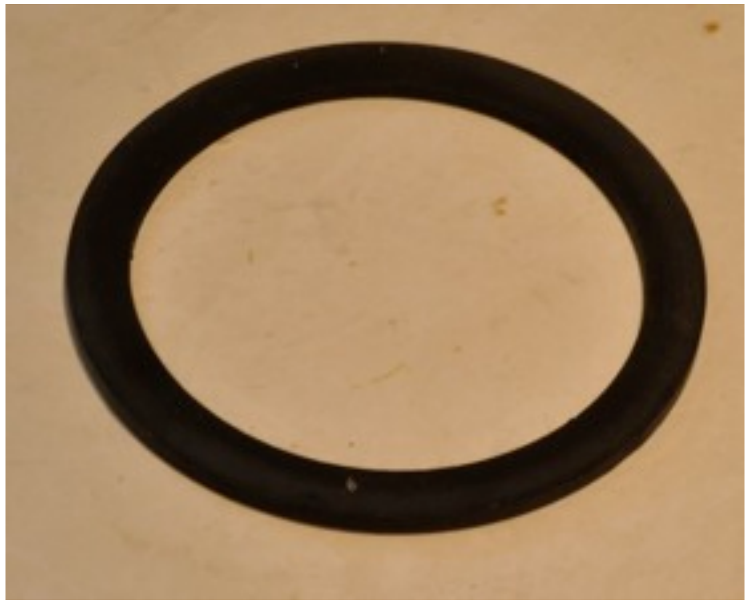
Loaded and residually stressed configuration

Virtual stress-free configuration

Ref: Decomposition introduced in elasto-plasticity (Kroner, ’60, Lee, ‘69), polymer swelling (’42), thermodynamics (Eckart ’48), Rodriguez et al (‘94).

1. A simple example

Back in Flatland
Consider an elastic ring in the plane



Grow the ring but only in the angular direction

"O day and night, but this is wondrous strange"

FLATLAND

A ROMANCE
OF MANY DIMENSIONS

By A Square

(Edwin A. Abbott)

"And therefore as a stranger give it welcome."

BASIL BLACKWELL · OXFORD

Price Seven Shillings and Sixpence net

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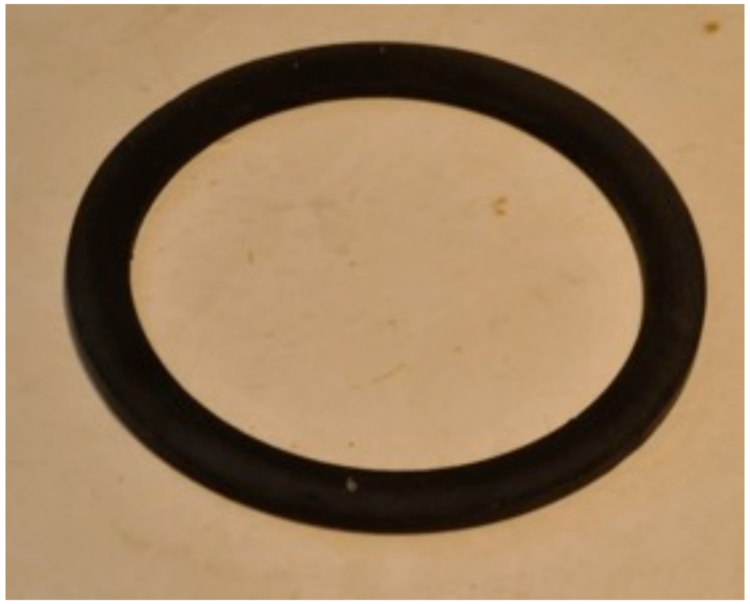

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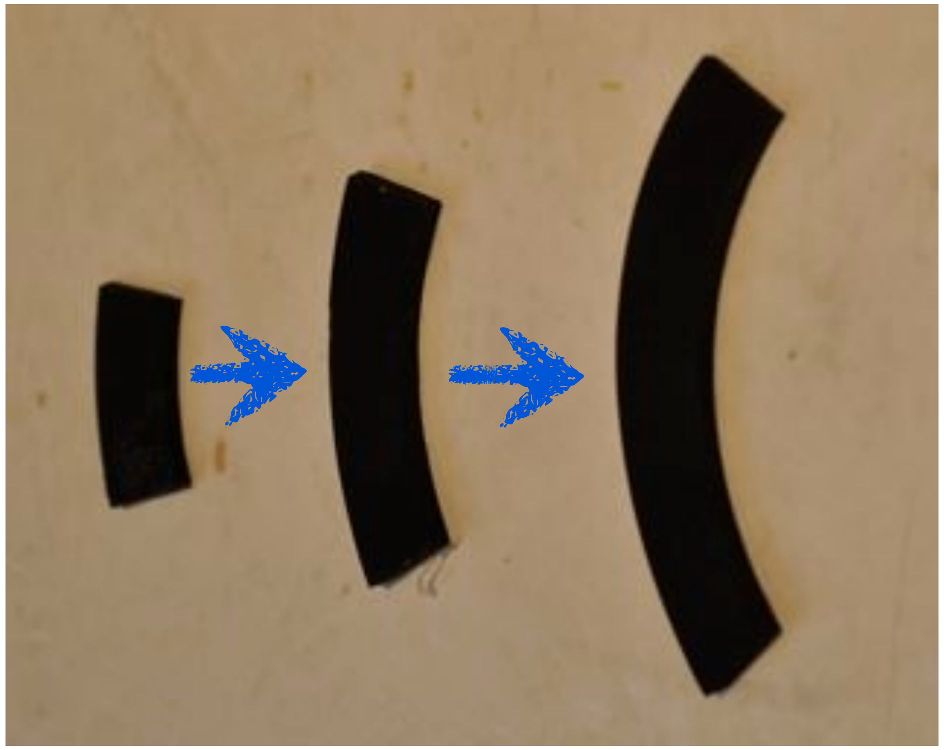

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$$\mathbf{G} = \begin{pmatrix} g_r & 0 \\ 0 & g_\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & g_\theta \end{pmatrix}$$

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Grown ring is stressed.

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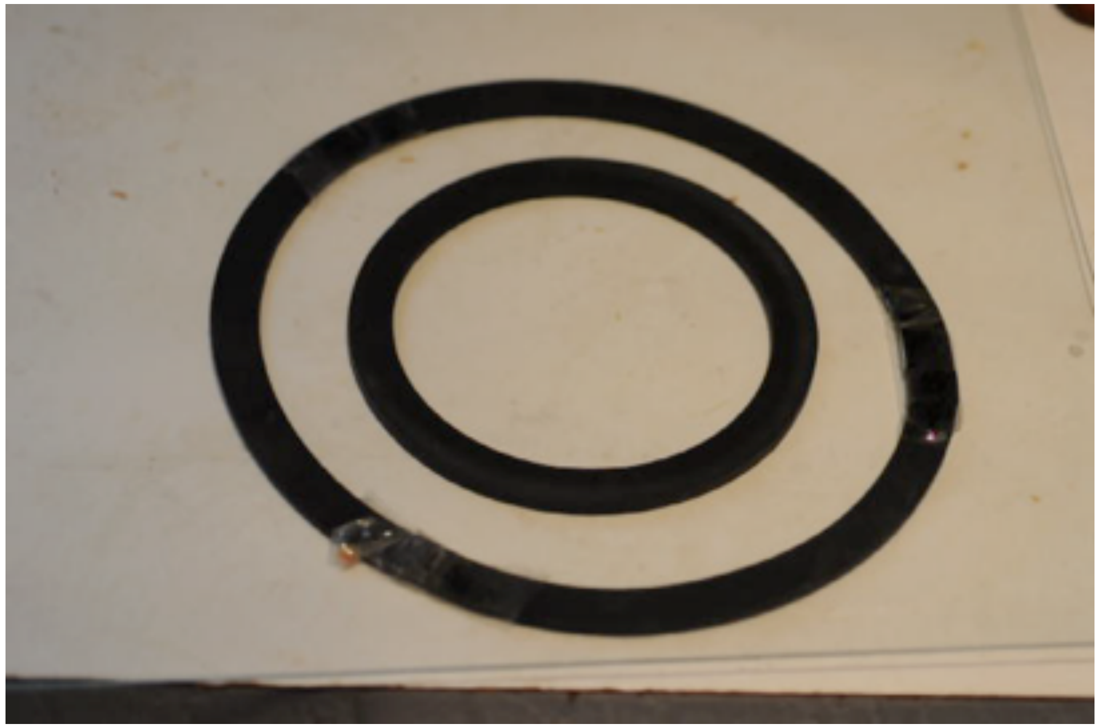

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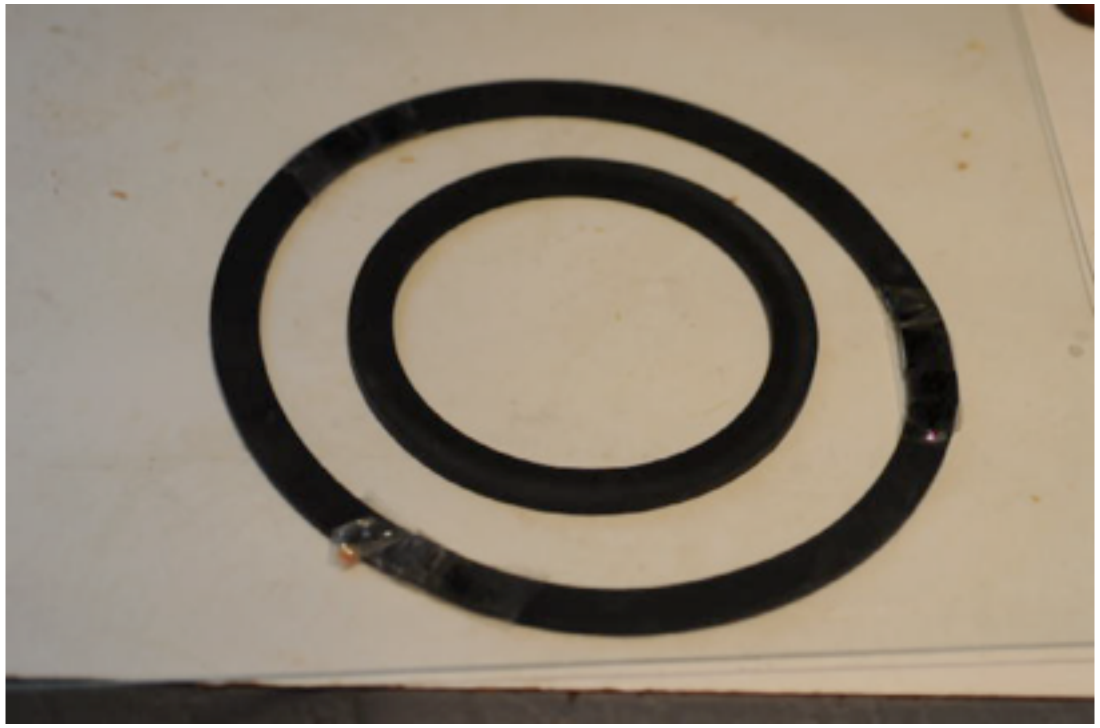

We can view it as a non-flat three dimensional object

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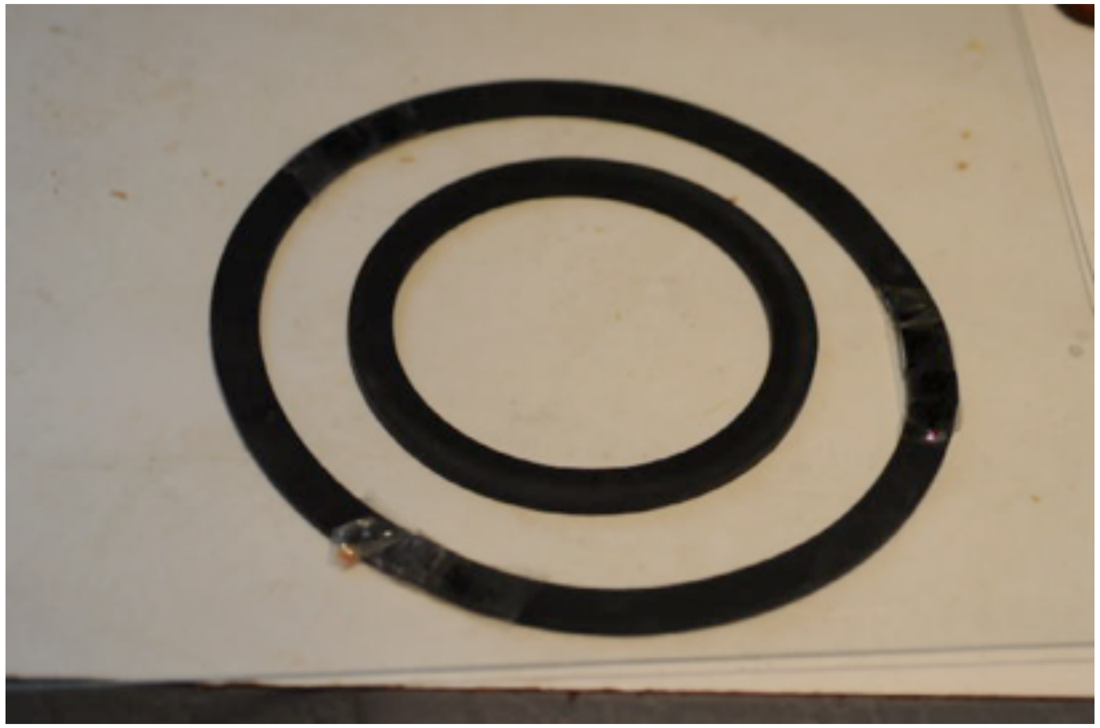

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We can view it as multivalued cut ring

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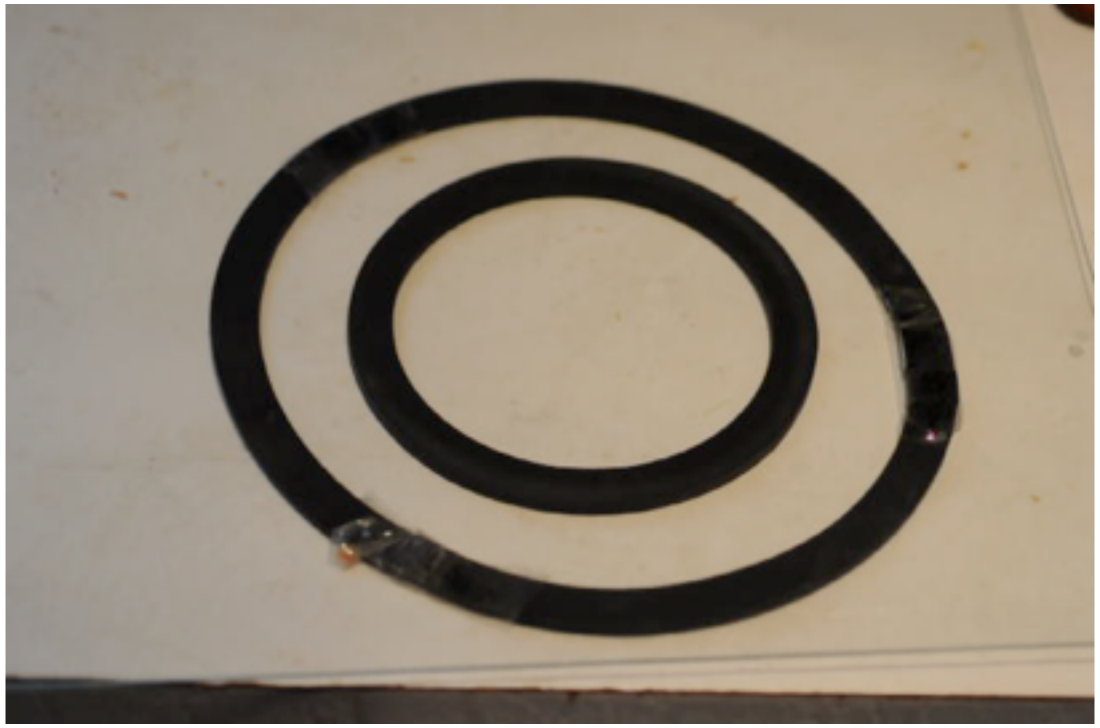

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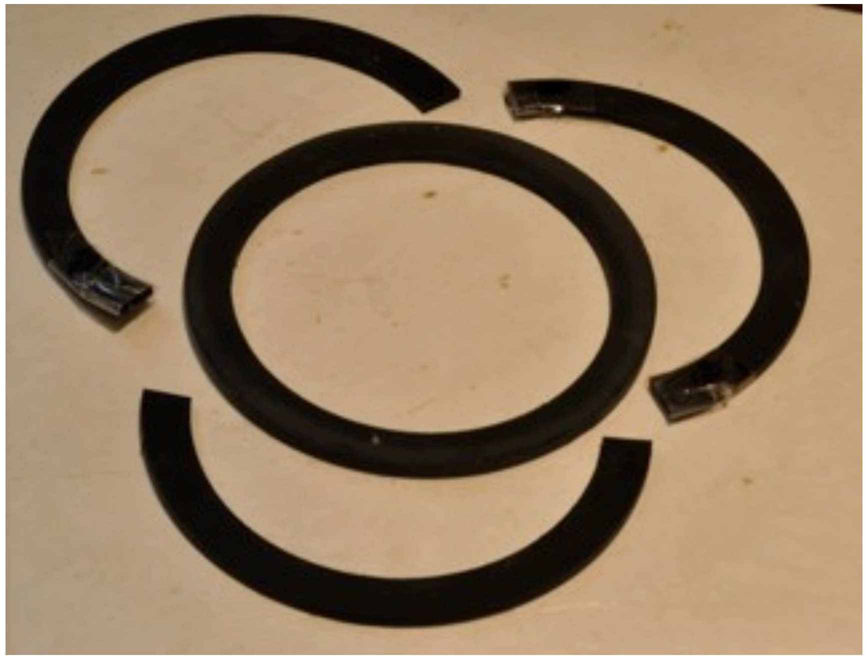
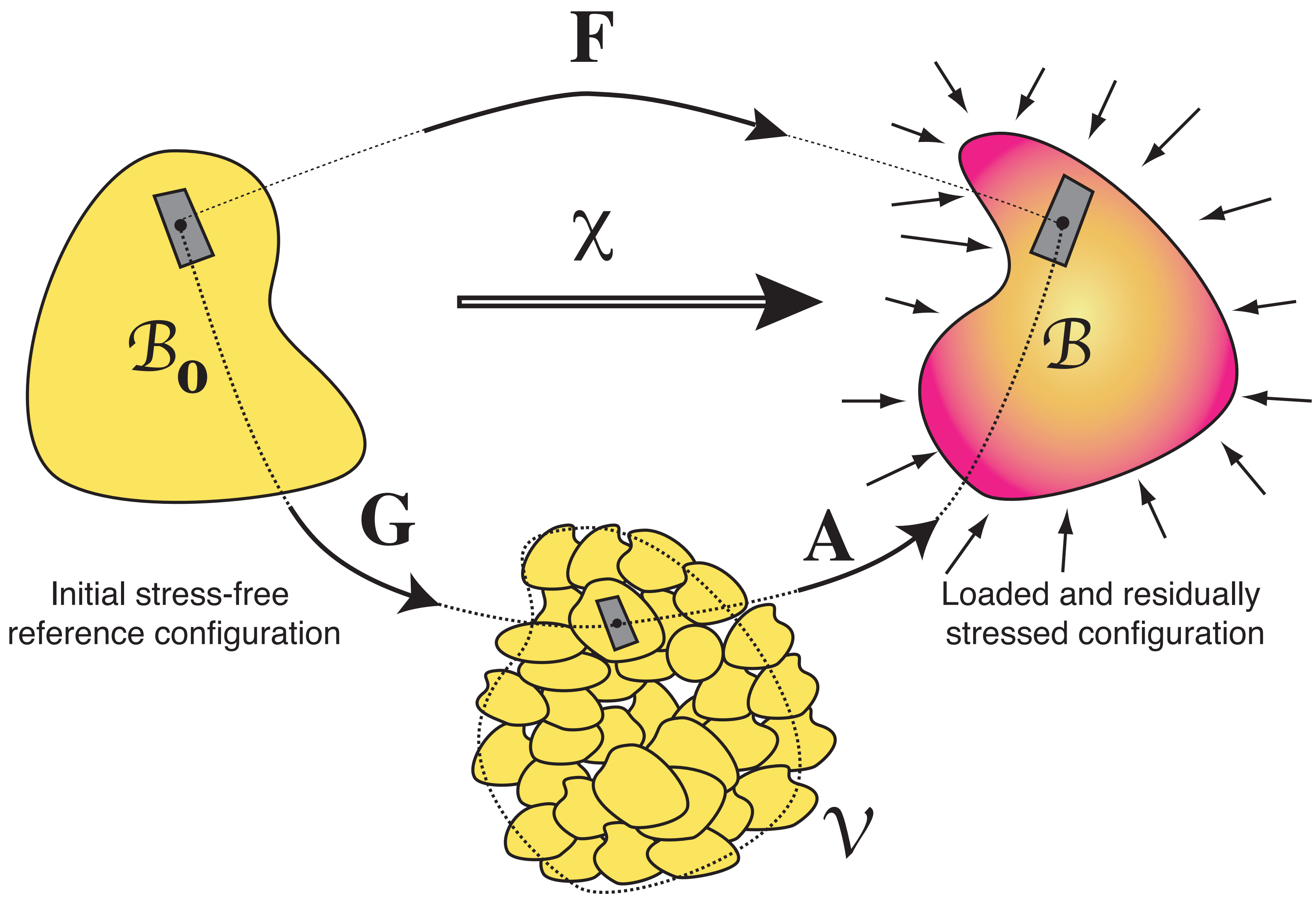
We can view it as multivalued cut ring

We can view it as collection of pieces



1. Kinematics

Decomposition: **F=AG** **G**: Growth deformation **A**: elastic deformation.



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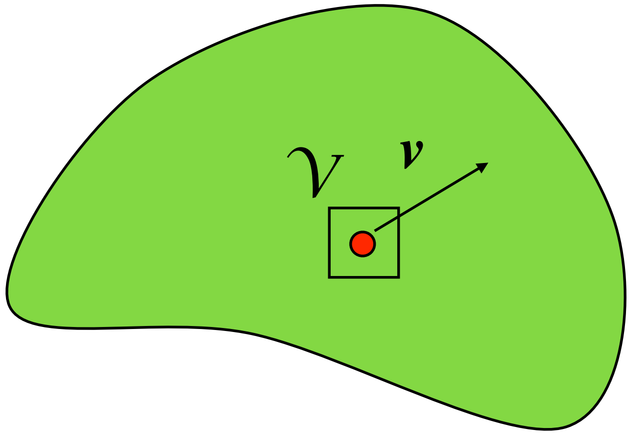
2. Mechanics

- Kinematics

  Deformation tensor $\mathbf{F}=\mathbf{Grad}(\chi)$



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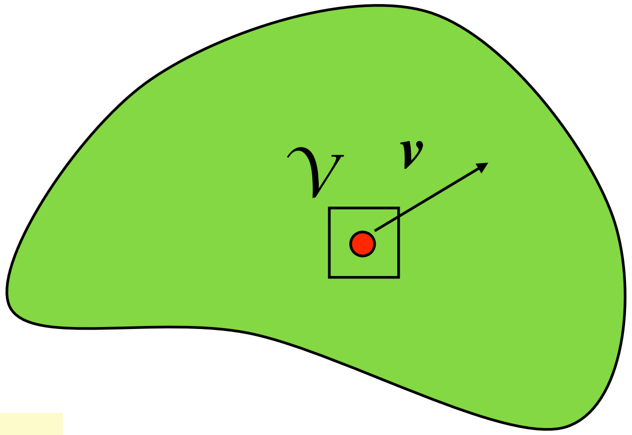

- Balance of mass

$$\frac{d\rho}{dt} + \rho\,\mathrm{div}(\mathbf{v}) = \pi + \mathrm{div}(\mathbf{m})$$

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- Kinematics

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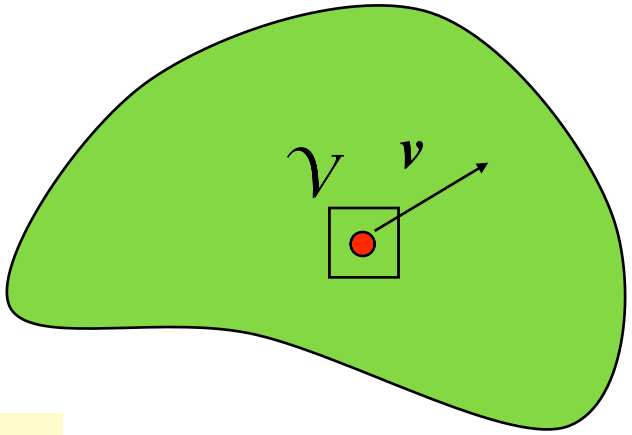

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$$\mathrm{div}(\mathbf{T}) + \rho\mathbf{b} = \rho\frac{d\mathbf{v}}{dt} - \mathbf{p} - \mathrm{div}(\mathbf{f}) - \mathbf{m}\cdot\mathrm{grad}(\mathbf{v})$$

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- Kinematics

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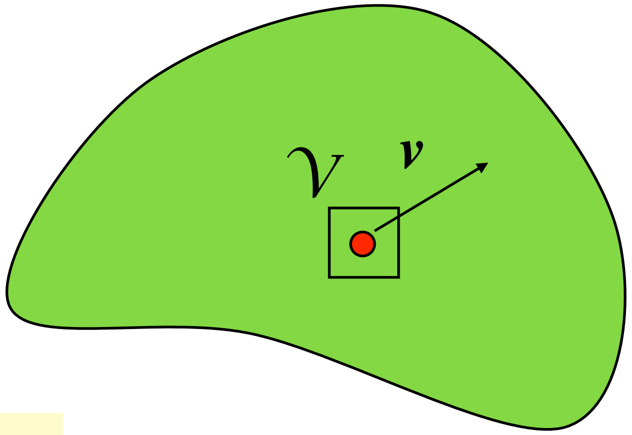

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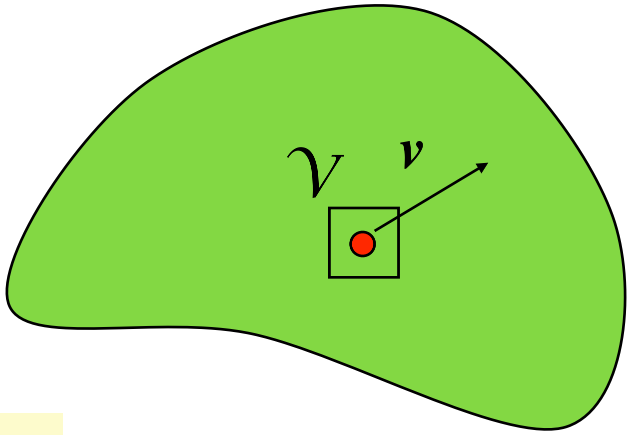

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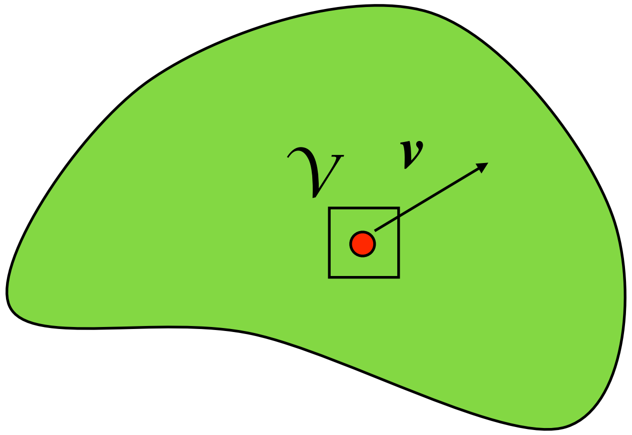

- Slow growth approximation

  Growth time scales much longer than forcing, viscous, or inertial terms.

- Balance of mass

$$\frac{d\rho}{dt} + \rho\,\mathrm{div}(\mathbf{v}) = 0$$

- Balance of linear momentum

$$\mathrm{div}(\mathbf{T}) + \rho\boldsymbol{B} = \rho\frac{d\mathbf{v}}{dt}$$

- Balance of angular momentum

$$\mathbf{T}^{\mathrm{T}} = \mathbf{T}$$

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  Growth time scales much longer than forcing, viscous, or inertial terms.

- Balance of mass

$$\frac{d\rho}{dt} + \rho \operatorname{div}(\mathbf{v}) = 0$$

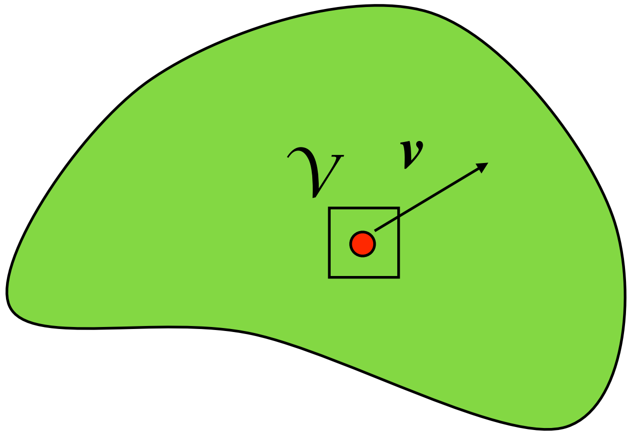
- Balance of linear momentum

$$\operatorname{div}(\mathbf{T}) + \rho \boldsymbol{B} = \rho \frac{d\mathbf{v}}{dt}$$

- Balance of angular momentum

$$\mathbf{T}^{\mathrm{T}} = \mathbf{T}$$

3. Elasticity



- Elasticity

  Constitutive relationship

$$\mathbf{T}=H(\mathbf{A})$$

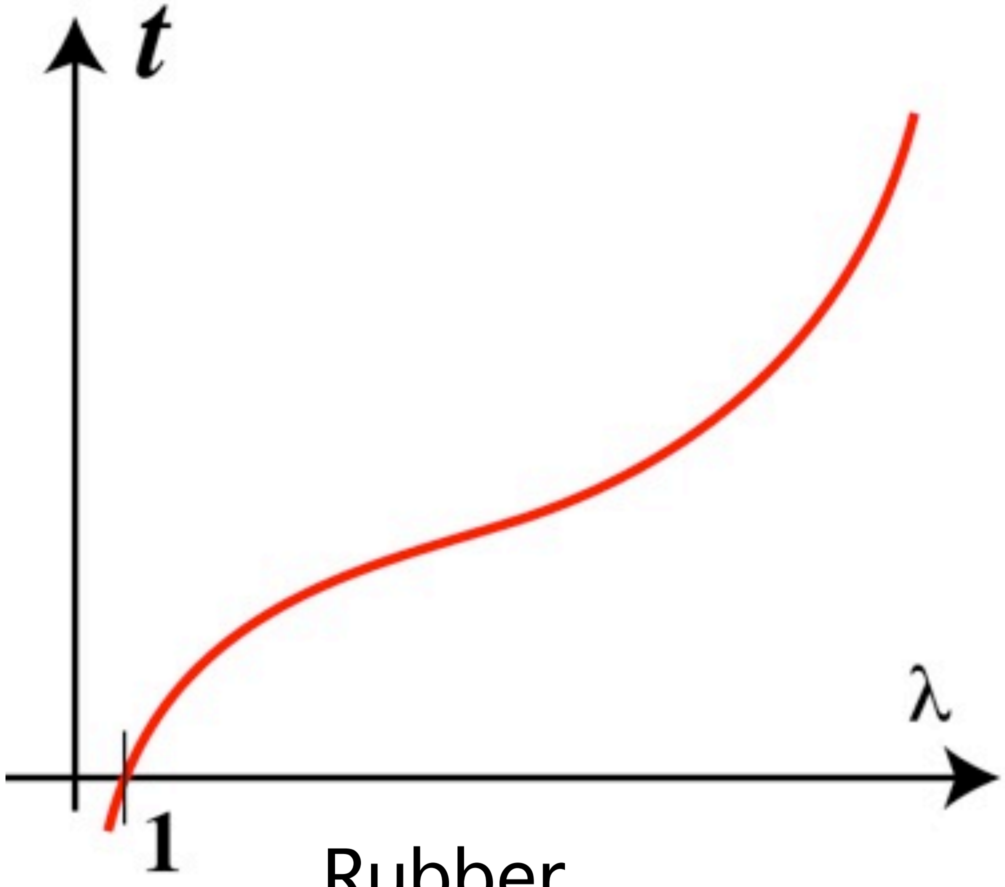
  Hyperelastic materials

  $W=W(\mathbf{A})$**:** *strain-energy density*

$$\mathbf{T} = \mathbf{A}\frac{\partial W}{\partial \mathbf{A}}(\mathbf{A})$$

3. Elasticity: Material response

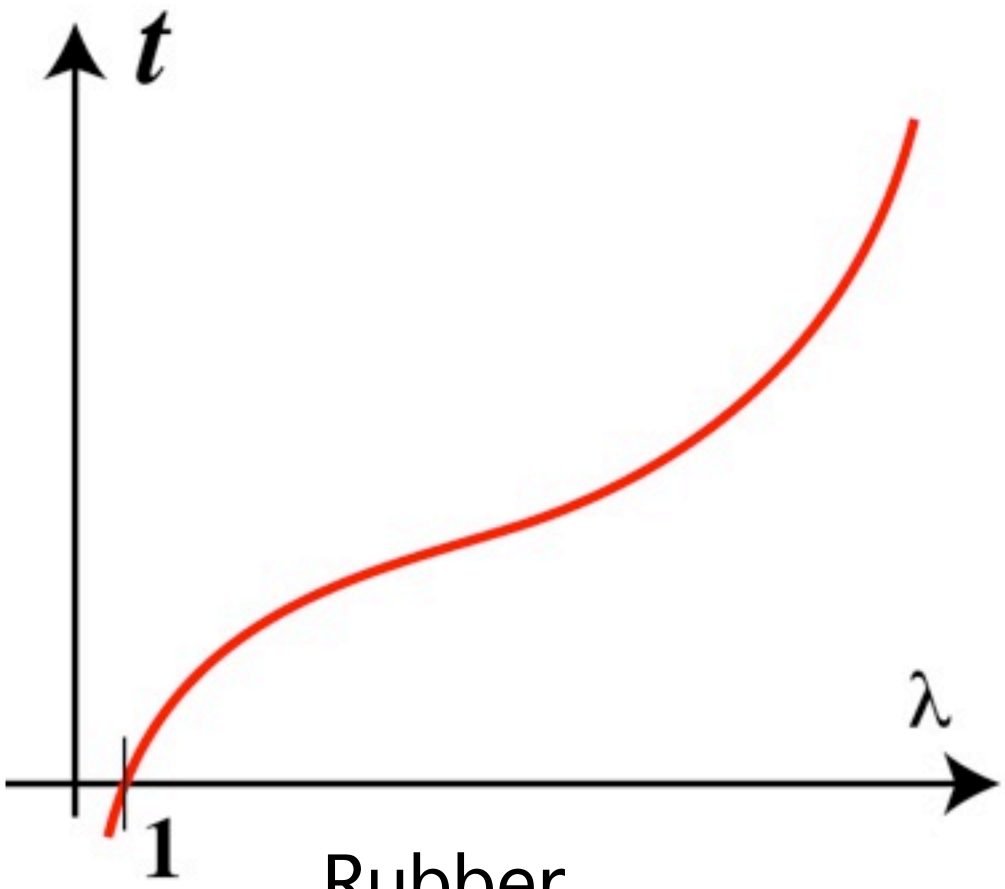
Neo-Hookean model



Rubber

3. Elasticity: Material response

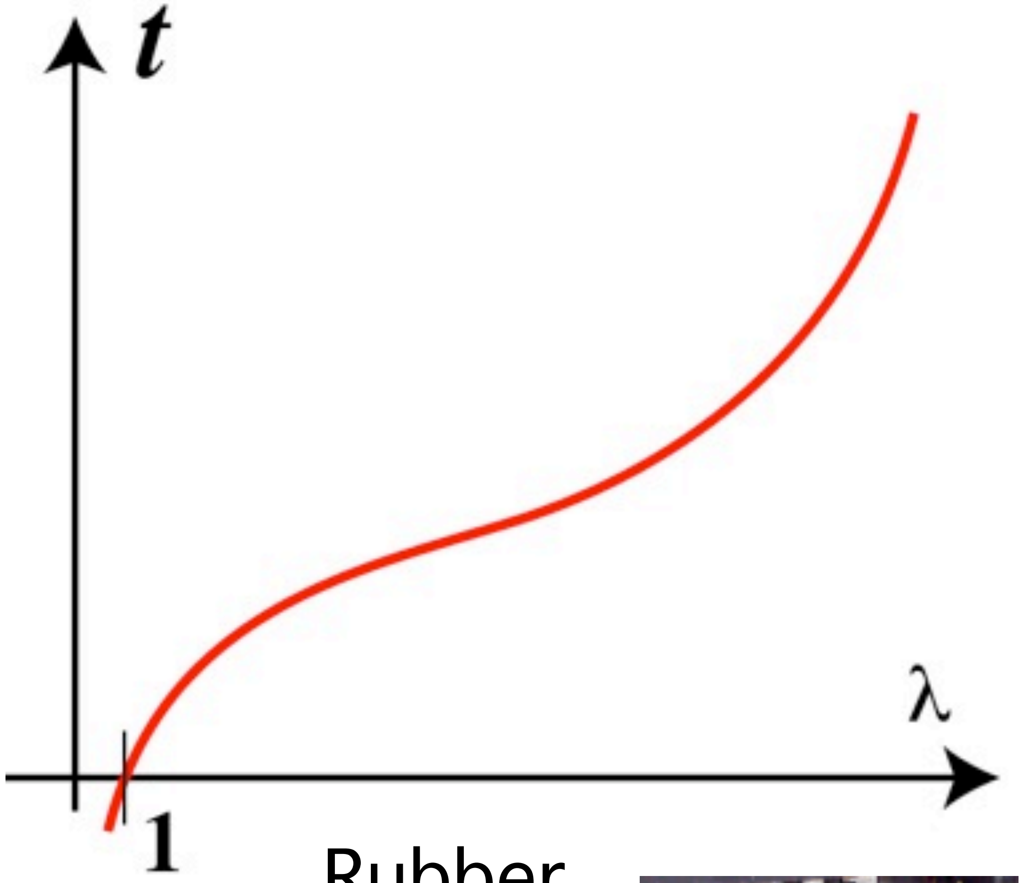
Neo-Hookean model



Rubber

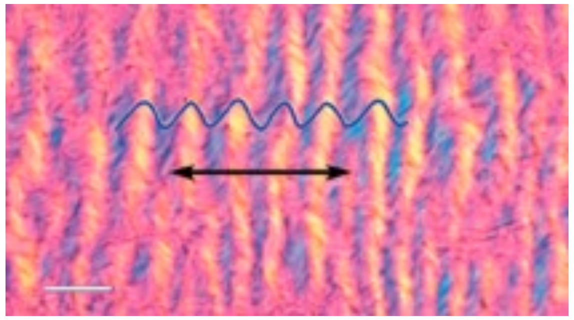
3. Elasticity: Material response

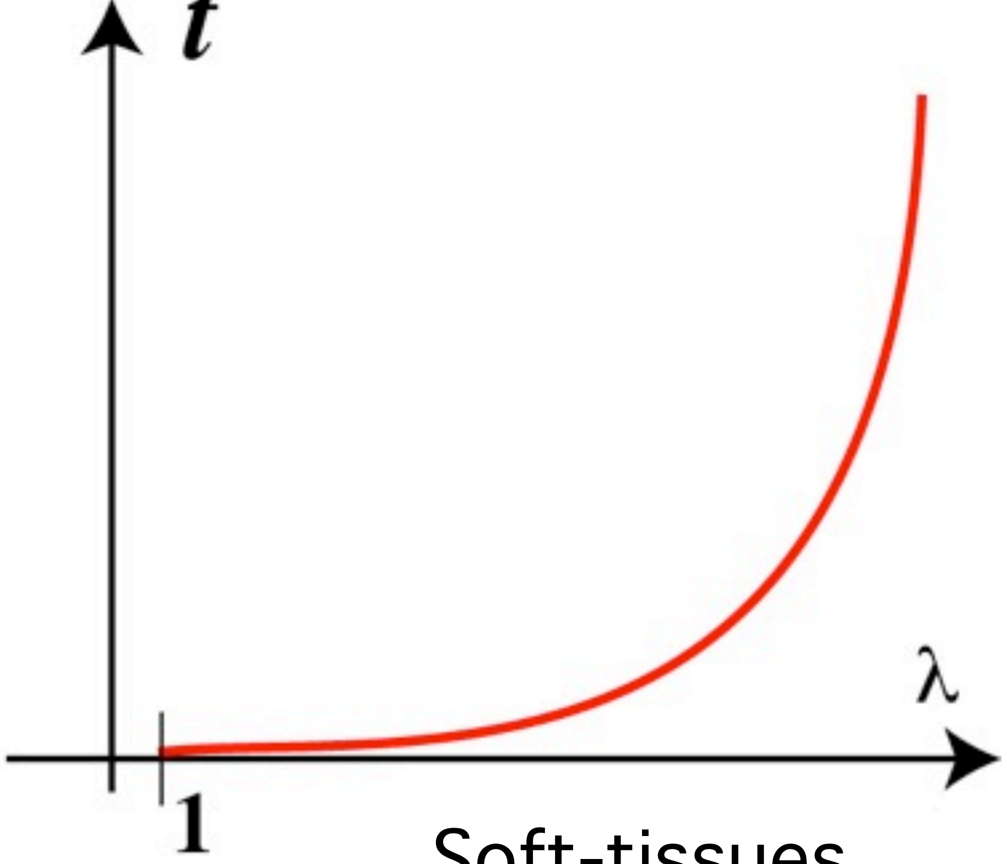
Neo-Hookean model



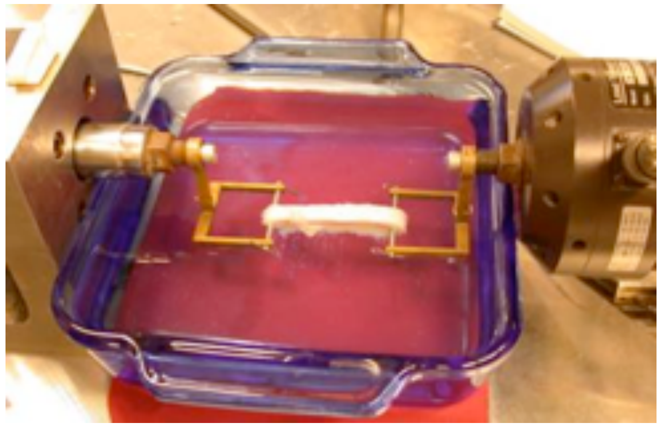
Rubber

Fung model





Soft-tissues



Sarah M. Wells,
Dalhousie U.

4. Stability

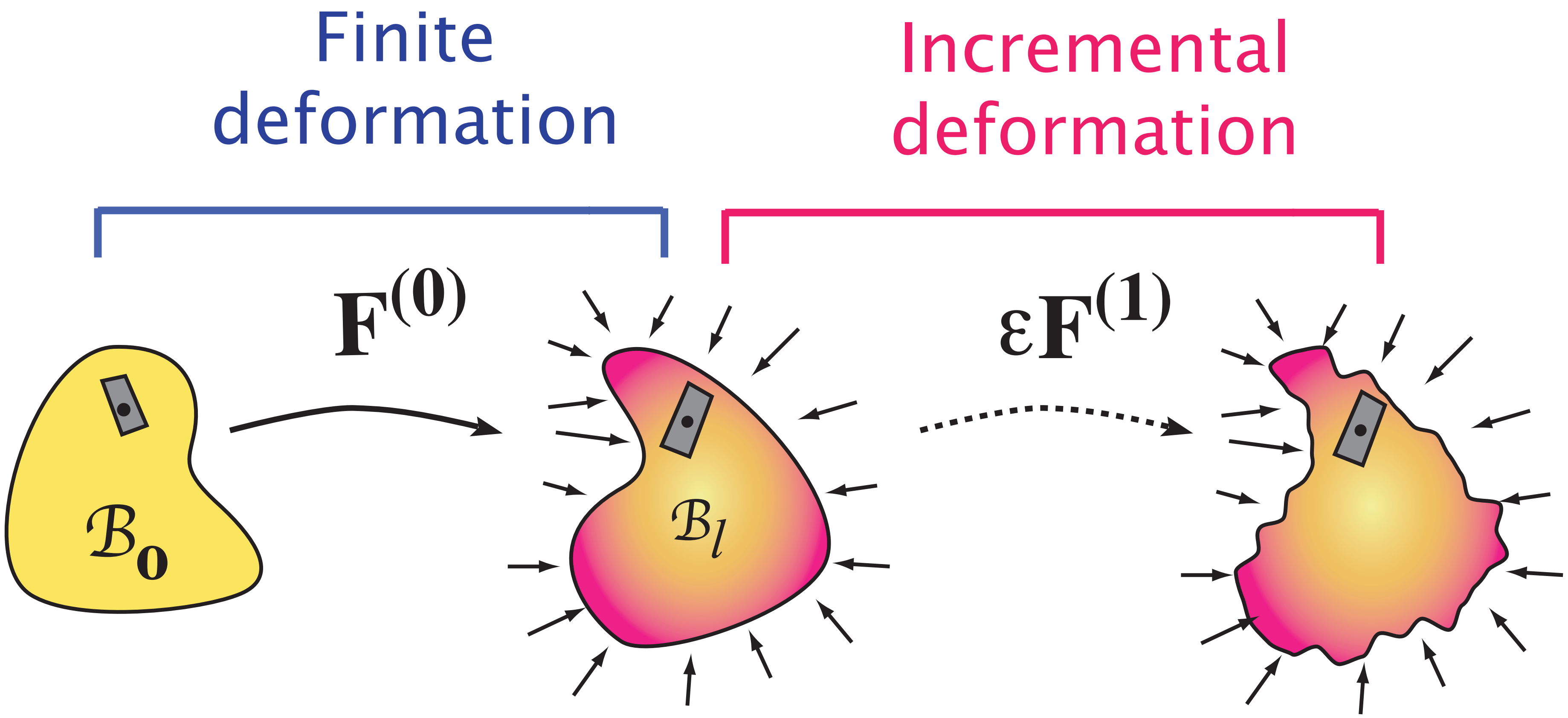
We consider incremental deformation superimposed on the finite deformation.

$$\boxed{\chi = \chi^{(0)} + \epsilon\chi^{(1)},}$$

Finite deformation

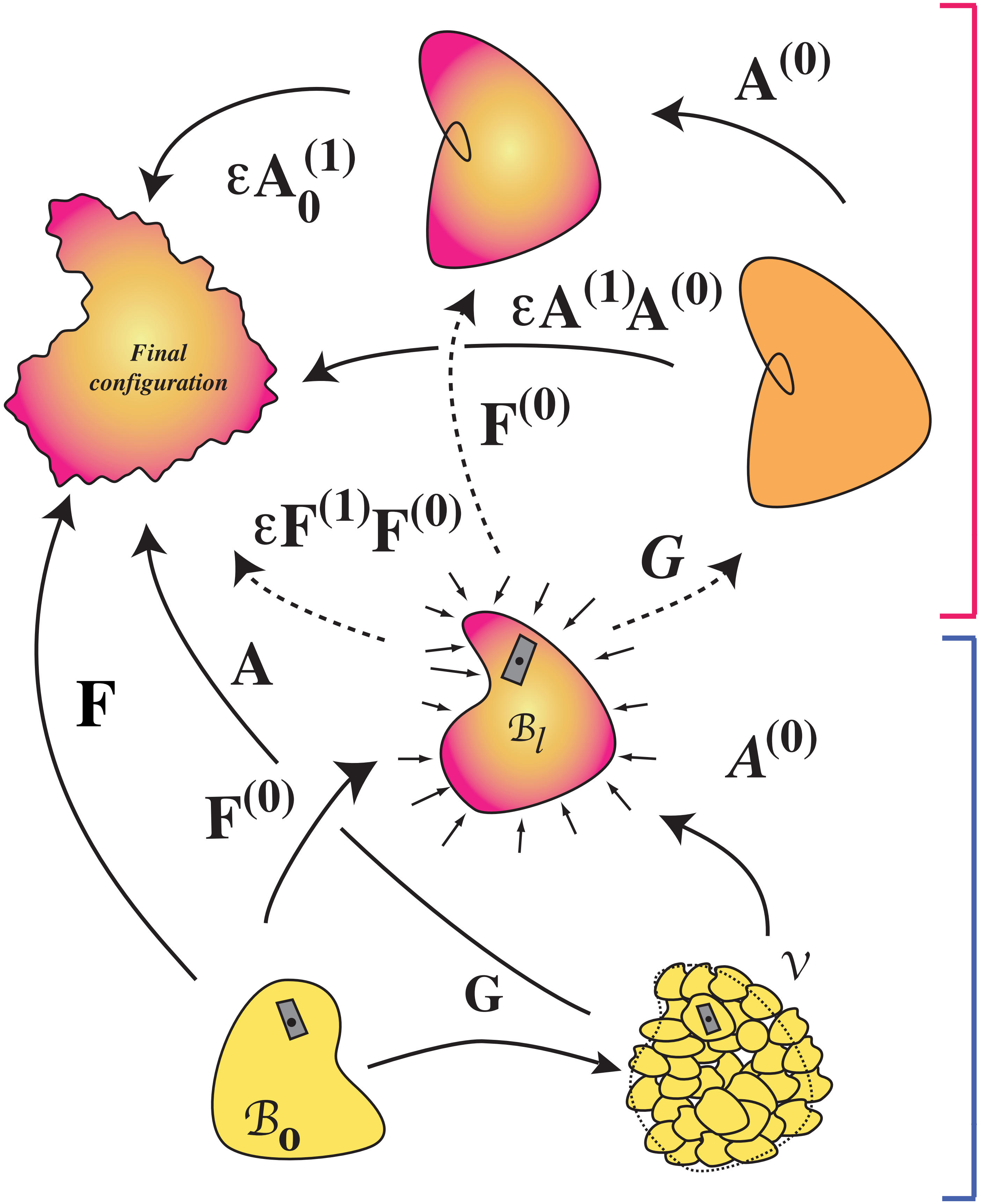
Incremental deformation ε<<1

The deformation tensor $\mathbf{F} = \mathrm{Grad}(\chi) = \left(\mathbf{1} + \epsilon\mathbf{F}^{(1)}\right) \cdot \mathbf{F}^{(0)}$


Finite deformation

Incremental deformation

4. Stability (Anelastic)



Incremental deformation

Finite deformation

4. Stability

The deformation tensor $\mathbf{F} = \mathrm{Grad}(\chi) = \left(\mathbf{1} + \epsilon \mathbf{F}^{(1)}\right) \cdot \mathbf{F}^{(0)}$

The Cauchy stress tensor $\mathbf{T} = \mathbf{T}^{(0)} + \epsilon \mathbf{T}^{(1)} + \mathrm{O}(\epsilon^2).$

To first order, the incremental stress tensor is

$$\boxed{\mathbf{T}^{(1)} = \mathcal{L} : \mathbf{F}^{(1)} + \mathbf{F}^{(1)} \cdot \mathbf{A}^{(0)} \cdot W_{\mathbf{A}}^{(0)} - p^{(1)} \mathbf{1}}$$

Instantaneous elastic moduli

Pressure

Second derivative of W w.r.t. A

where
$$\boxed{\mathcal{L} : \mathbf{F}^{(1)} = \mathbf{A}^{(0)} W_{\mathbf{A}\mathbf{A}}^{(0)} : \mathbf{F}^{(1)} \mathbf{A}^{(0)},}$$

The linearized equation is then

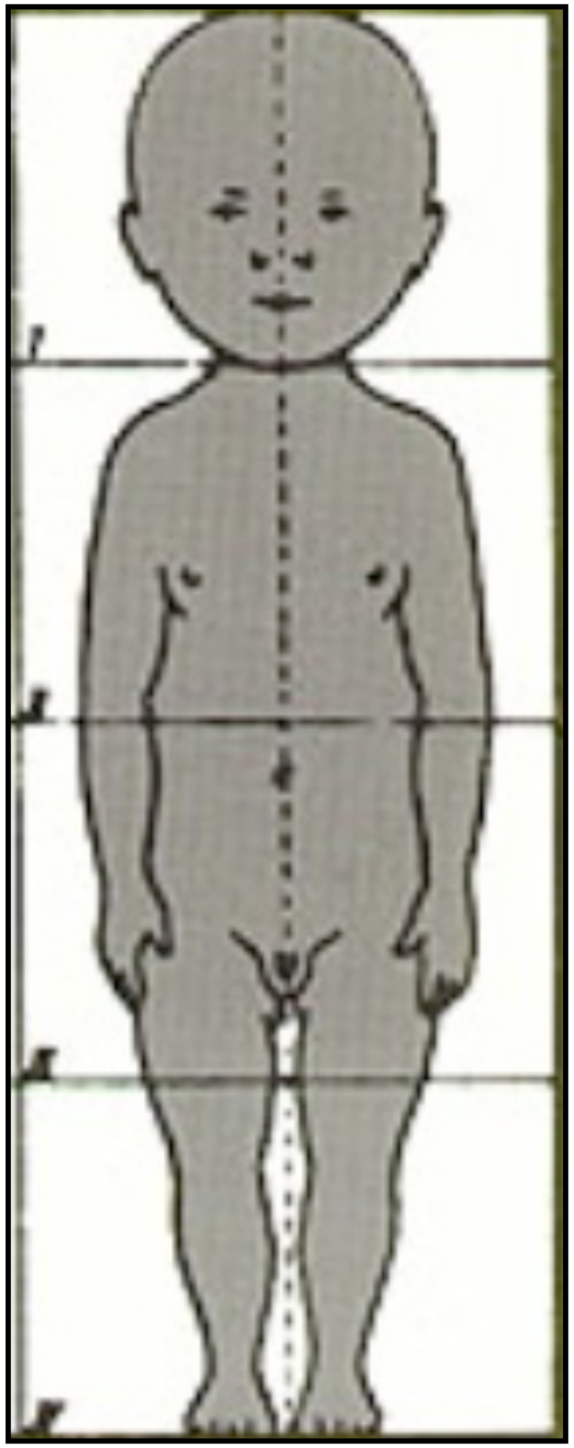
$$\boxed{\mathrm{div}(\mathrm{T}^{(1)}) = 0}$$

+ Boundary Conditions

Part 3. Analysis of growth

■ 3. Consequences

- ✤Dynamics
- ✤Bifurcation
- ✤Instability



Dynamics

Homogeneous growth

- Considering homogeneous diagonal deformations

$$\begin{aligned}
&\mathbf{F} = \mathbf{A}\mathbf{G}\\
&\dot{\mathbf{G}} = \mathcal{E}(\mathbf{T},\mathbf{A})\\
&\mathcal{H}(\mathbf{T},\mathbf{A}) = 0\\
&\mathcal{F}(\mathbf{T},\mathbf{F},t) = 0
\end{aligned}$$

Kinematics	3
Growth law	3
Response function	3
Boundary conditions	3

$$\begin{aligned}
&\dot{\mathbf{G}} = \mathcal{E}(\mathbf{T},\mathbf{A})\\
&\frac{\partial\mathcal{H}}{\partial\mathbf{T}}\dot{\mathbf{T}} + \frac{\partial\mathcal{H}}{\partial\mathbf{A}}\dot{\mathbf{A}} = 0\\
&\frac{\partial\mathcal{F}}{\partial\mathbf{T}}\dot{\mathbf{T}} + \frac{\partial\mathcal{F}}{\partial\mathbf{F}}\dot{\mathbf{A}}\mathbf{G} = -\frac{\partial\mathcal{F}}{\partial\mathbf{F}}\mathbf{A}\mathcal{E}(\mathbf{T},\mathbf{A}) - \partial_t\mathcal{F}
\end{aligned}$$

➔Systems of 9 ODEs for the tensors **G, A, T** (24 in general)

➔Predict the evolution of stresses, growth, and strains.

Dynamics: example

$$\dot{\mathbf{G}} = \mathcal{E}(\mathbf{T}, \mathbf{A})$$

$$\frac{\partial\mathcal{H}}{\partial\mathbf{T}}\dot{\mathbf{T}} + \frac{\partial\mathcal{H}}{\partial\mathbf{A}}\dot{\mathbf{A}} = 0$$

$$\frac{\partial\mathcal{F}}{\partial\mathbf{T}}\dot{\mathbf{T}} + \frac{\partial\mathcal{F}}{\partial\mathbf{F}}\dot{\mathbf{A}}\mathbf{G} = -\frac{\partial\mathcal{F}}{\partial\mathbf{F}}\mathbf{A}\mathcal{E}(\mathbf{T}, \mathbf{A}) - \partial_t\mathcal{F}$$

Dynamics: example

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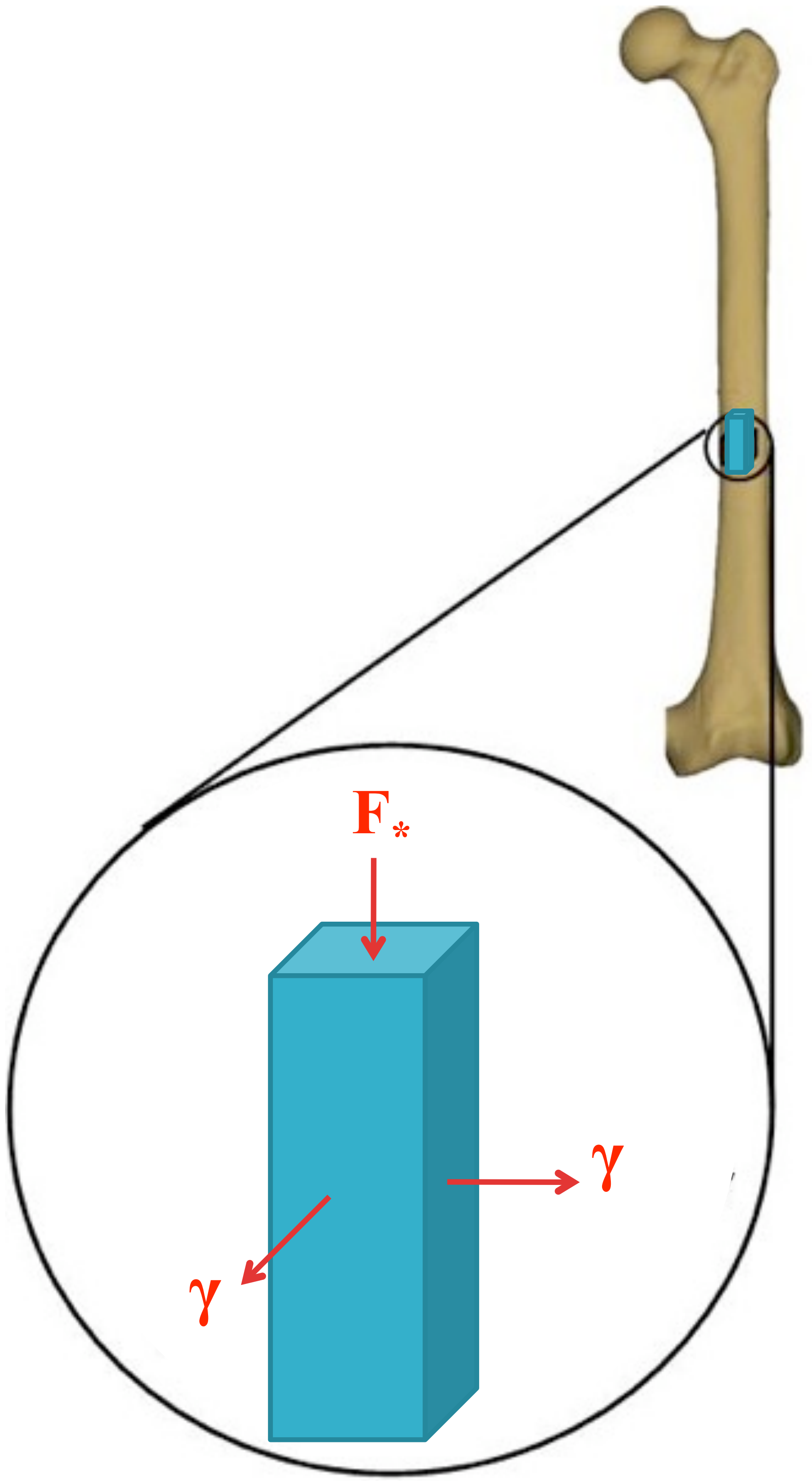


Dynamics: example

$$\dot{\alpha} = -\frac{\alpha^4 k T_z (T_z - T^*)}{\gamma(-4\mu\gamma - 2\mu\gamma\alpha^6 + T_z\alpha^3)}$$

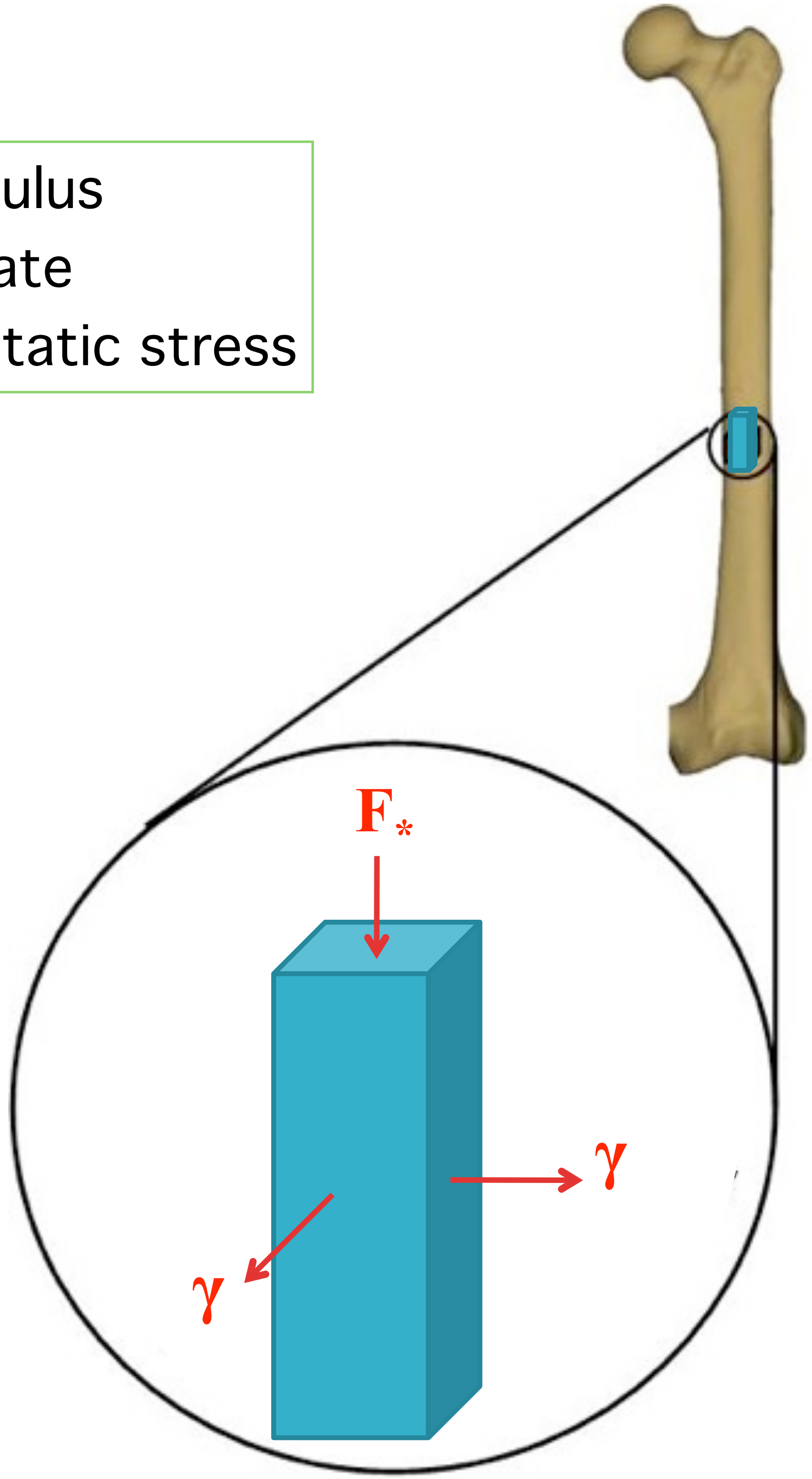
$$\dot{\gamma} = k(T_z - T^*)$$

$$\dot{T}_z = \frac{4\mu(2+\alpha^6)kT_z(T_z - T^*)}{\alpha\gamma(-4\mu\gamma - 2\mu\gamma\alpha^6 + T_z\alpha^3)}$$

μ=E/6: Modulus

K: growth rate

T*: Homeostatic stress



Dynamics: example

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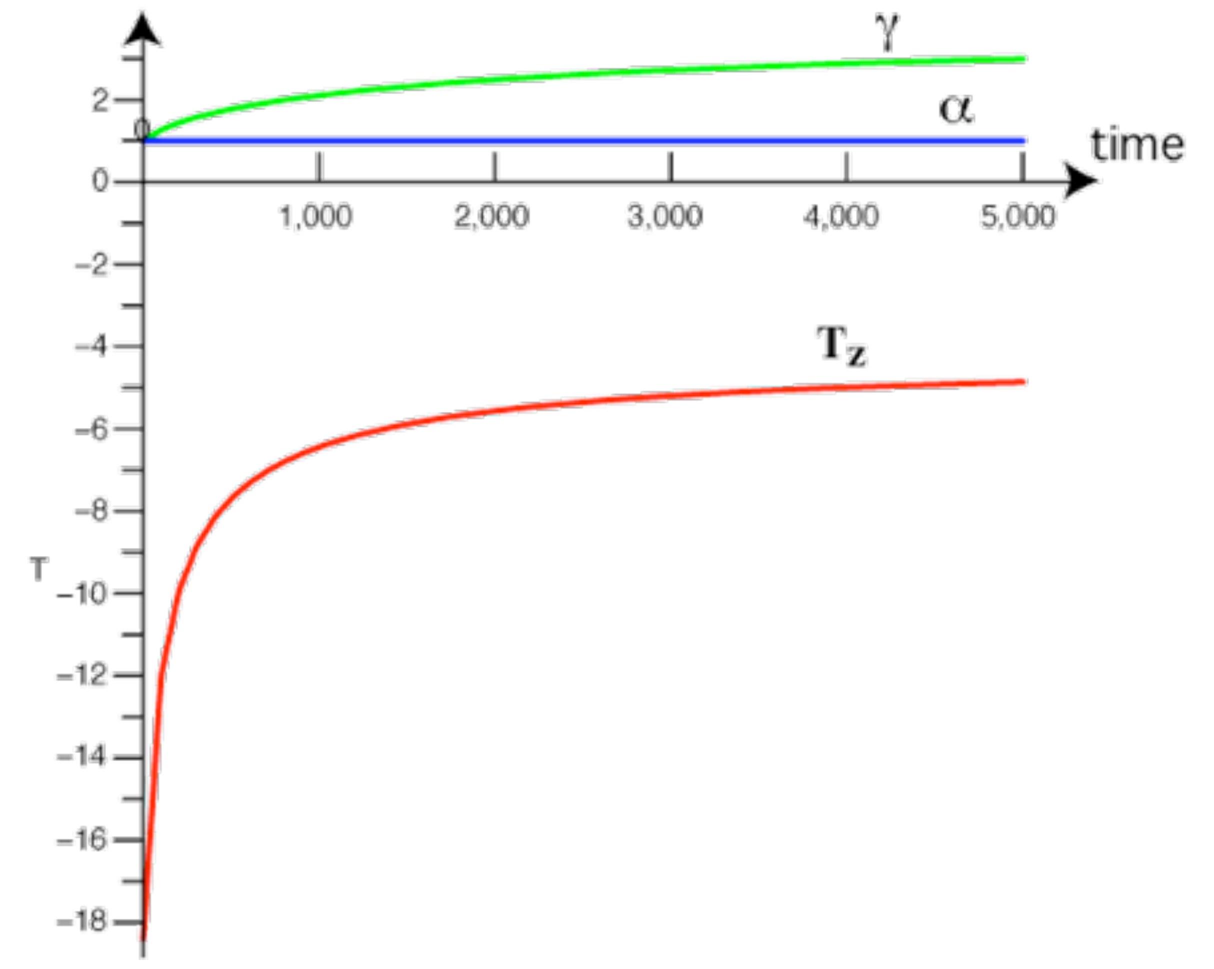
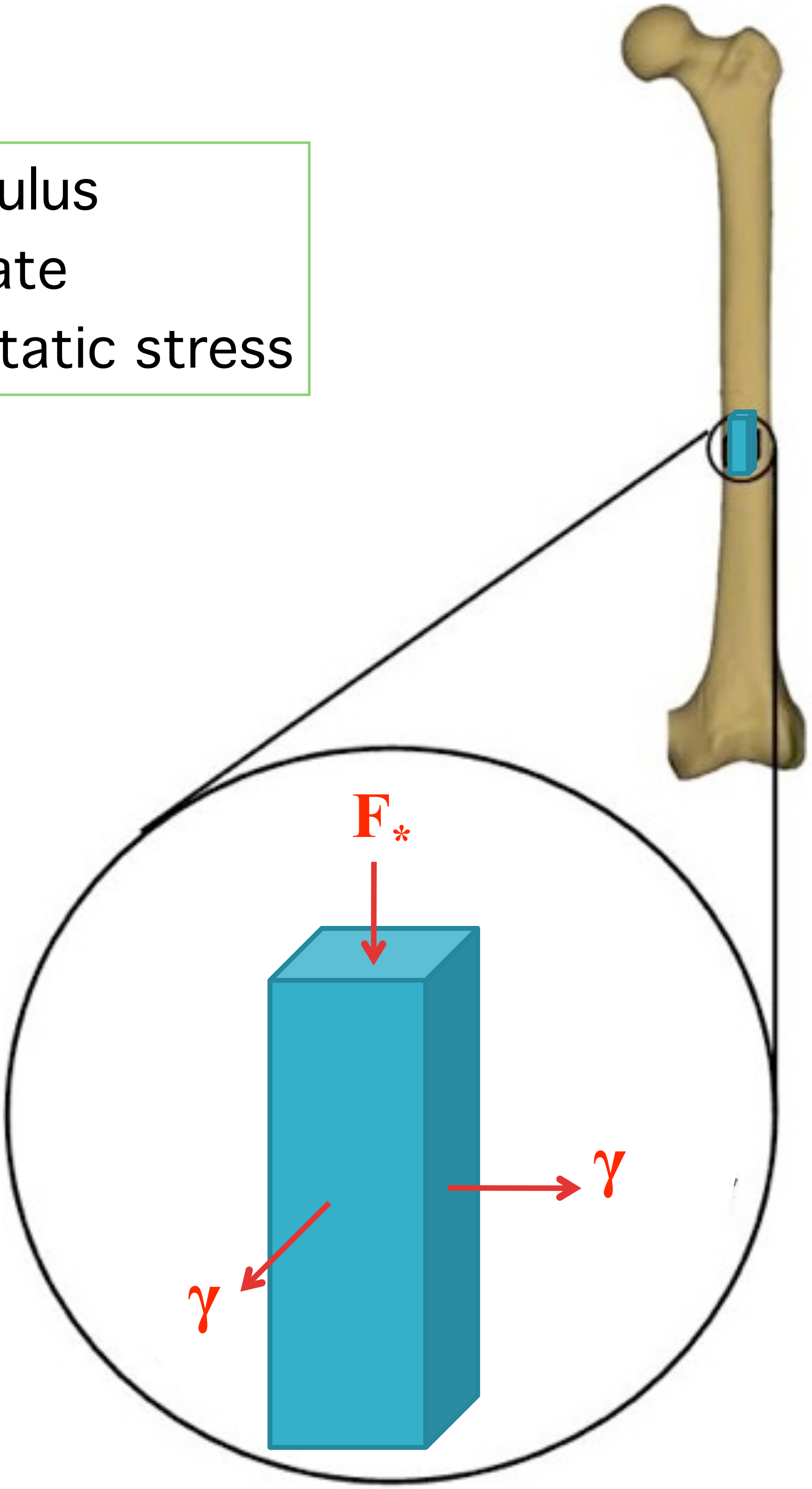
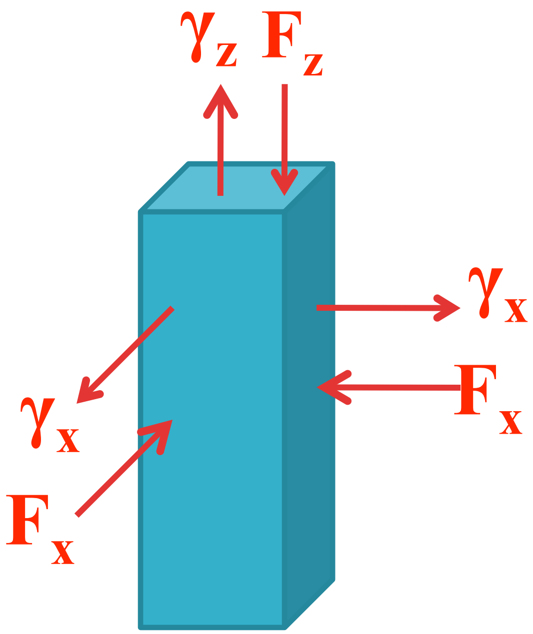




Dynamics: example

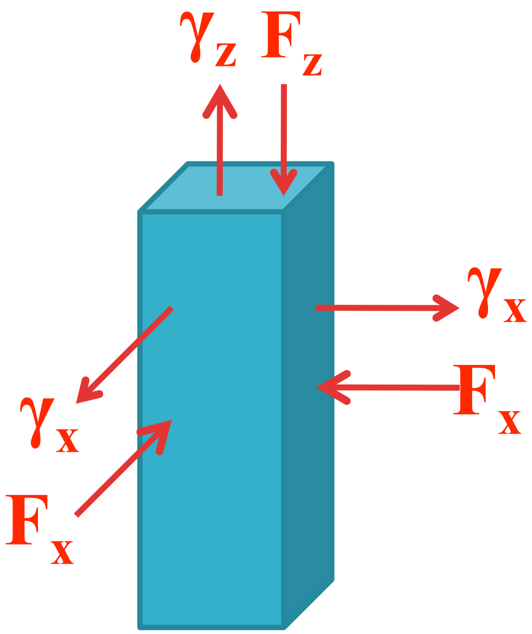
A two-dimensional example



$$\begin{pmatrix} \dot{\gamma}_x \\ \dot{\gamma}_z \end{pmatrix} = \mathbf{K} \begin{pmatrix} T_x - T_x^* \\ T_z - T_z^* \end{pmatrix}$$

Dynamics: example

A two-dimensional example



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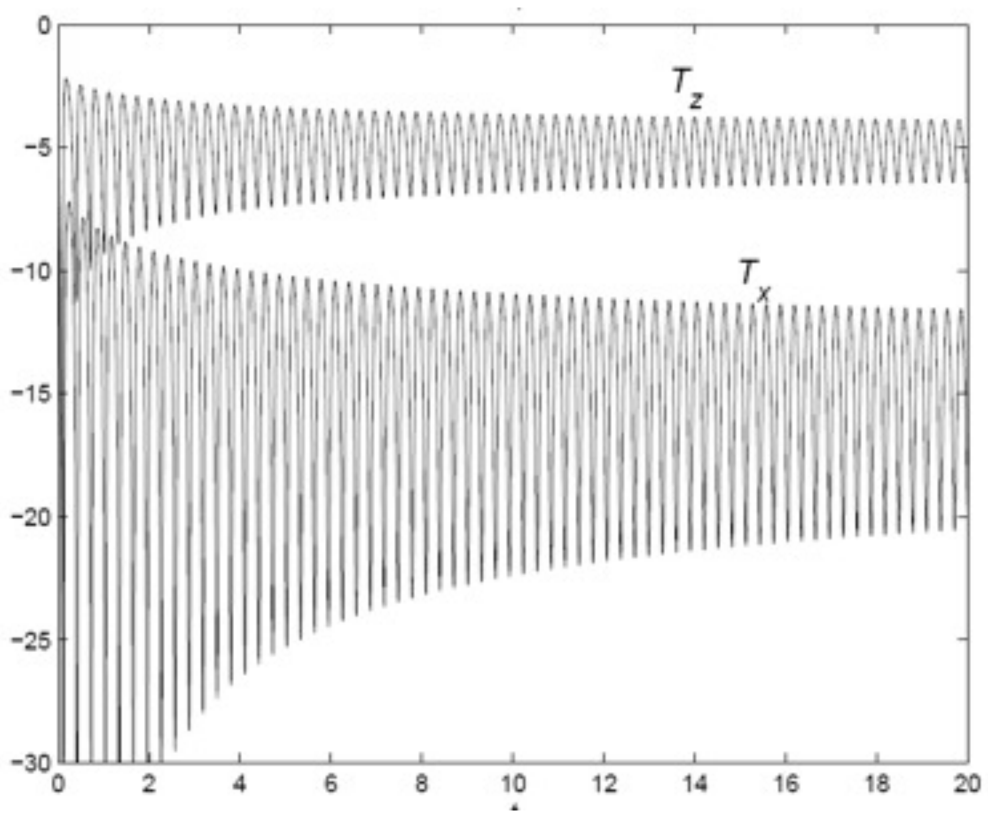
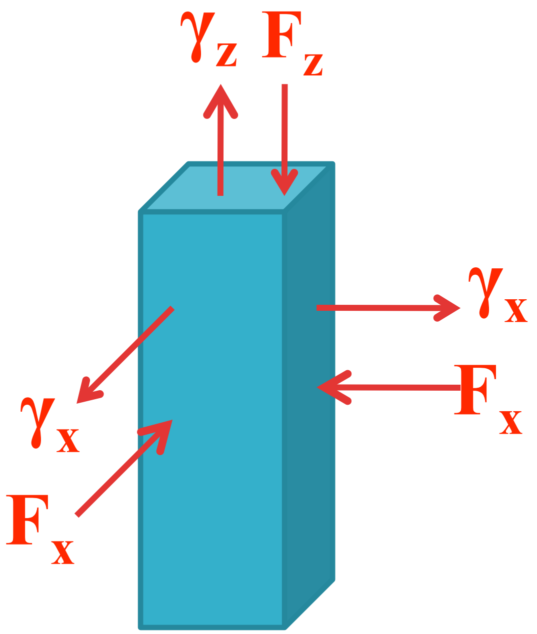


Dynamics: example

A two-dimensional example



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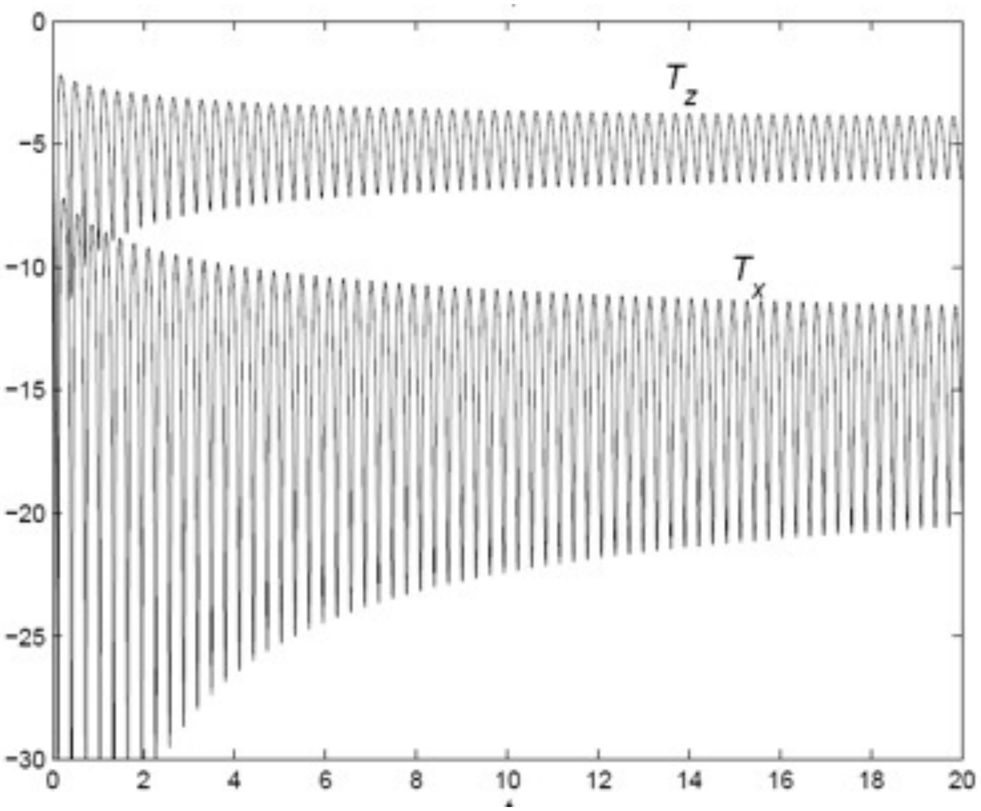
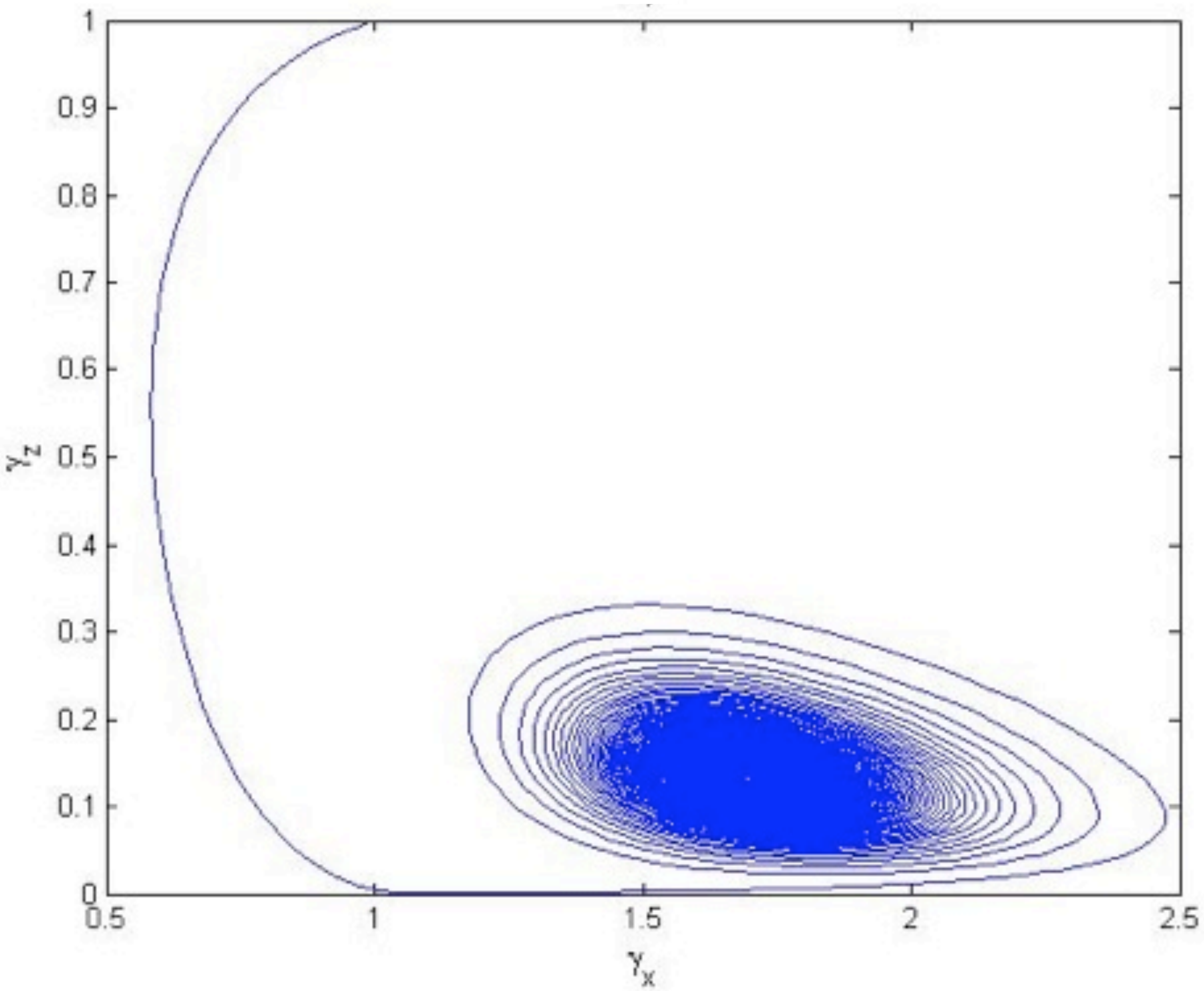
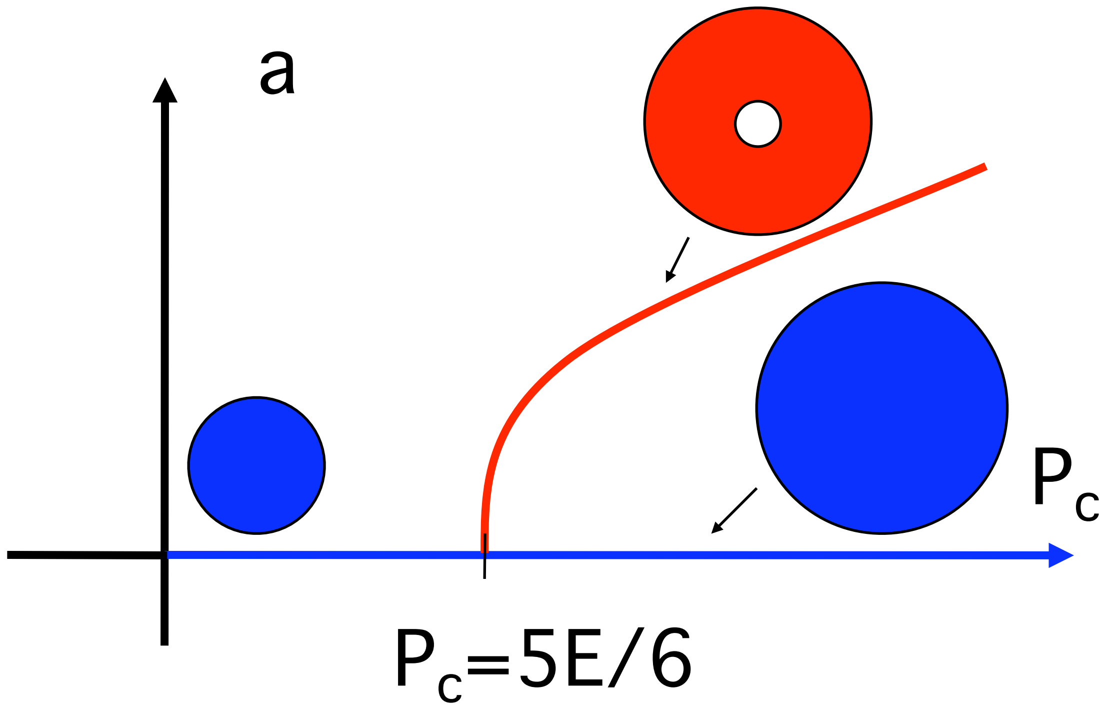
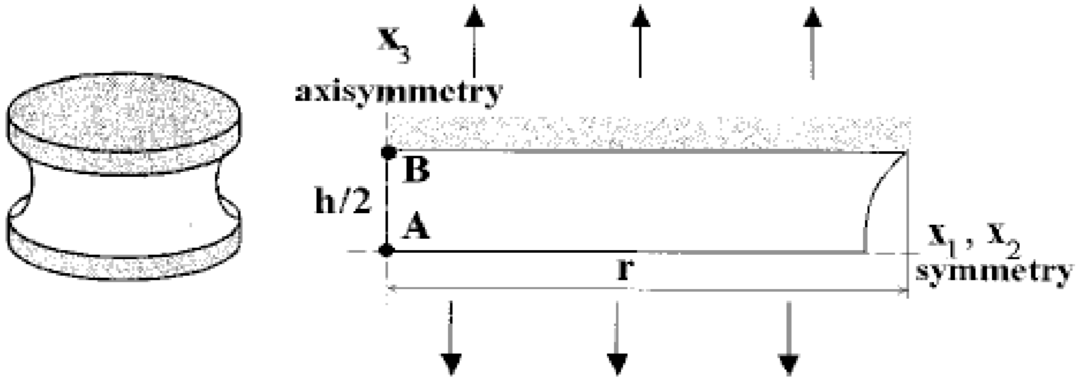
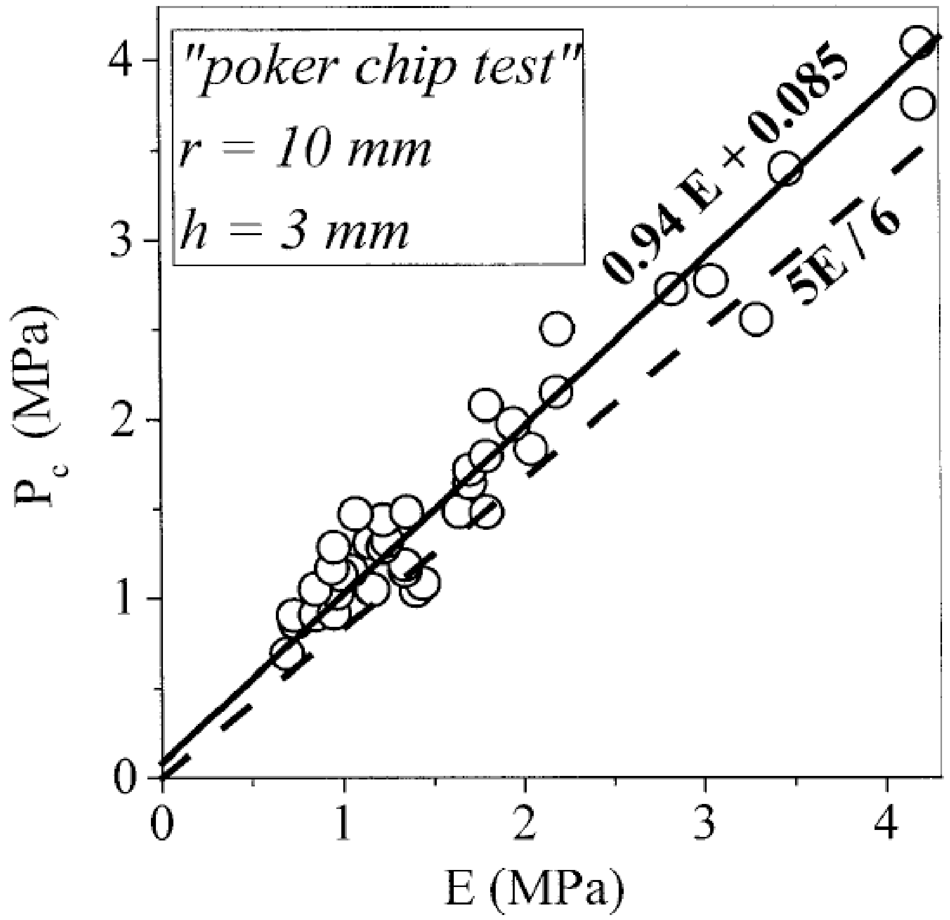




Bifurcation: Cavitation



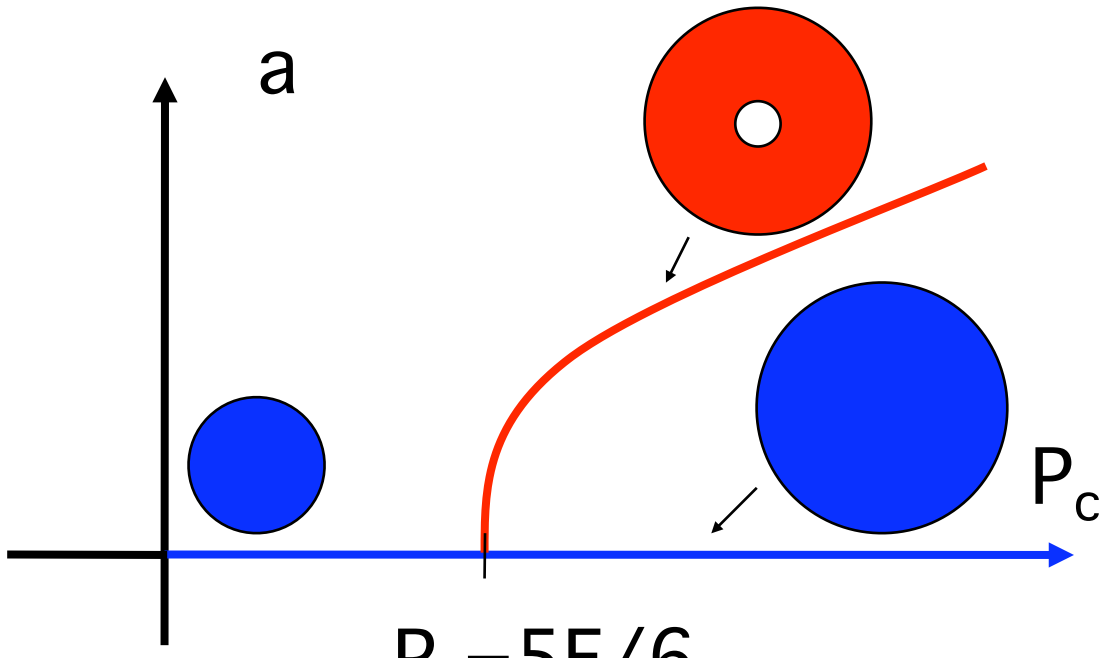




Fond 2001 Int. J. Polym. Sci. B

Ref: Ball ('82)

Bifurcation: Cavitation

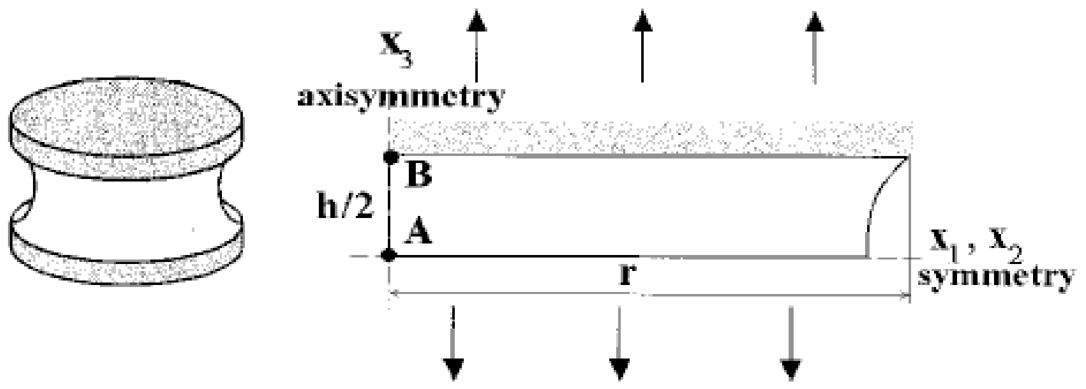


$P_c=5E/6$

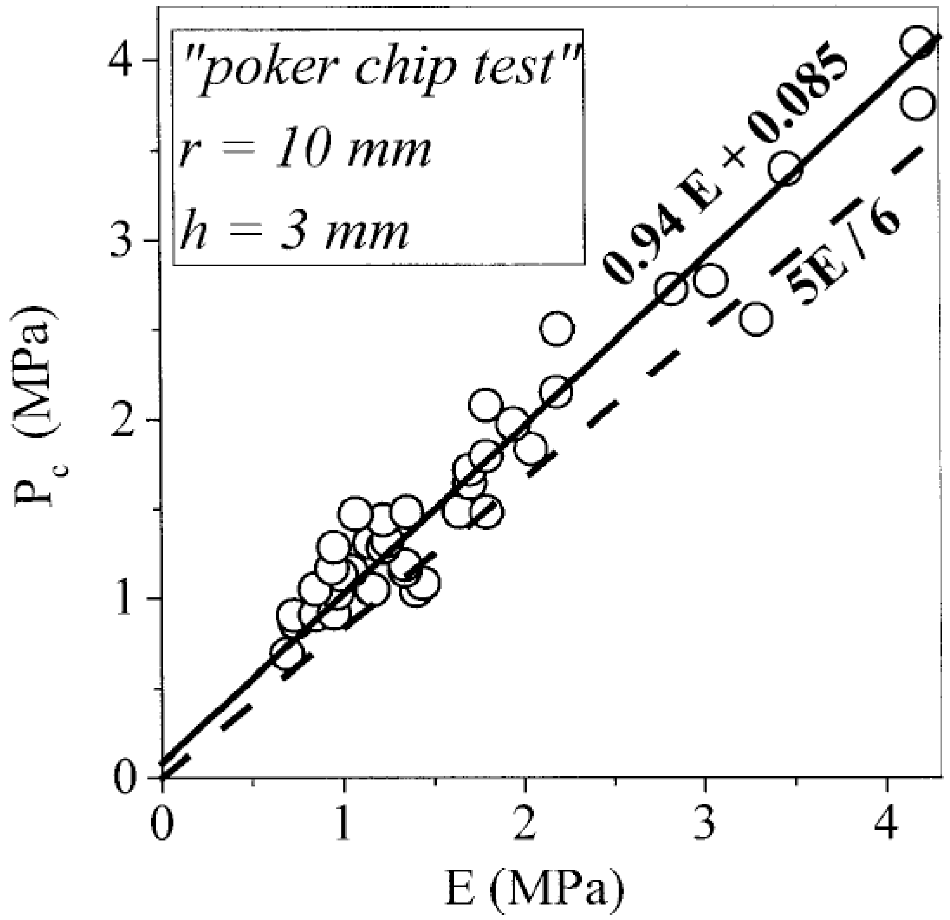
Problems:

1. Topologically unstable

$W=W_{nh}+\varepsilon\,\lambda^n$ with $n>2$



➔



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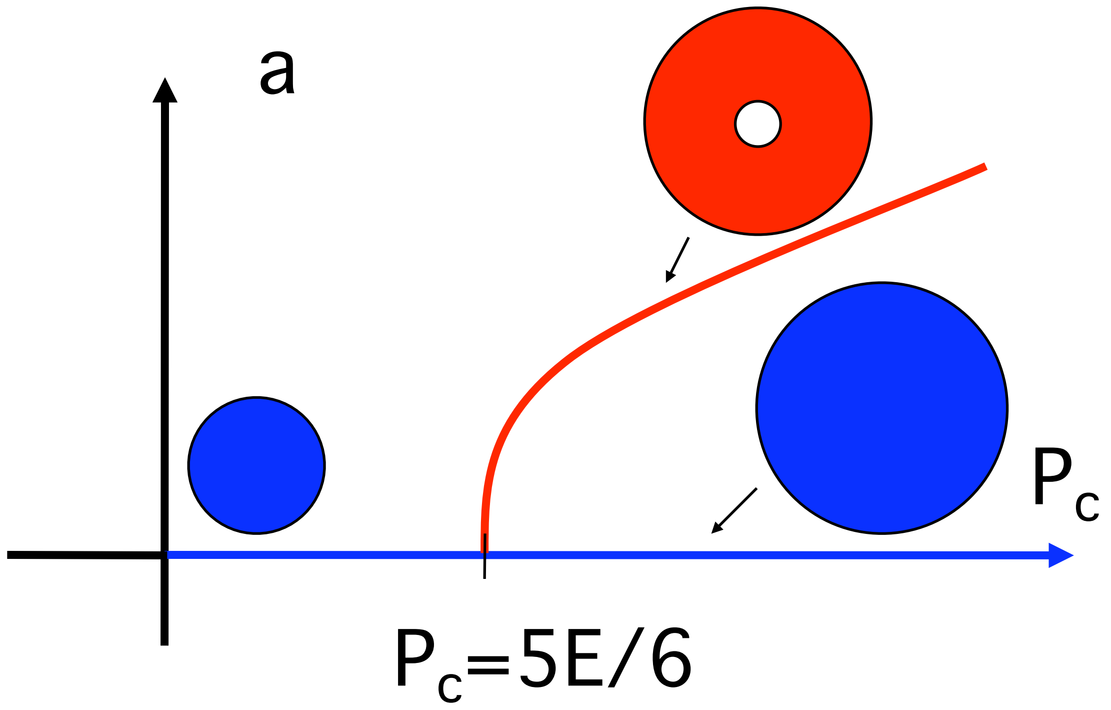
Bifurcation: Cavitation

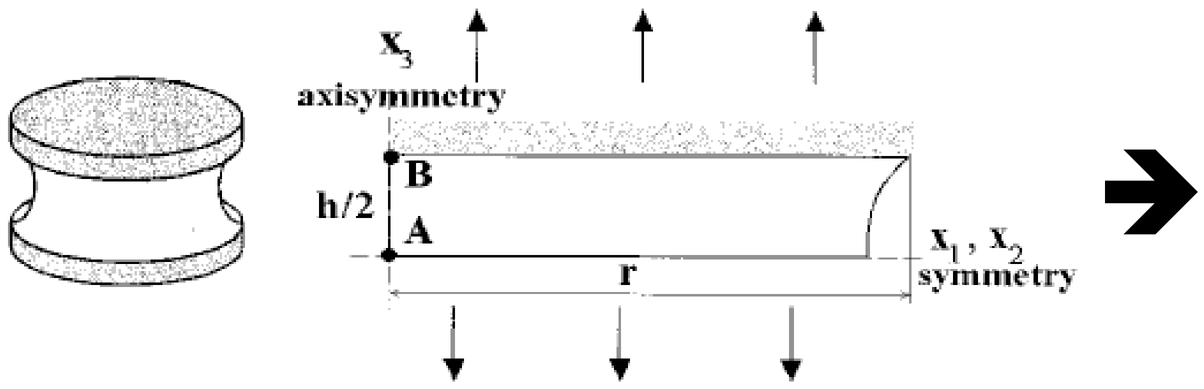


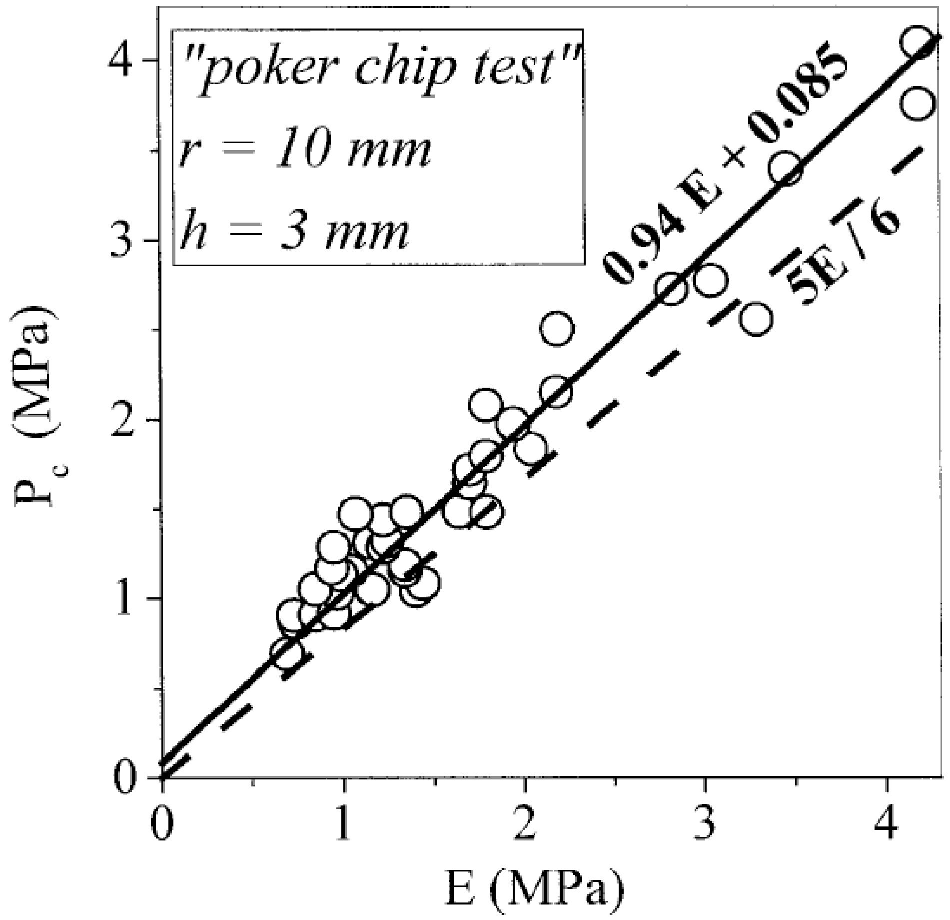
Problems:

1. Topologically unstable

   $W=W_{nh}+\varepsilon\,\lambda^n$ with $n>2$

2. No cavitation in neo-Hookean cylinder





Fond 2001 Int. J. Polym. Sci. B

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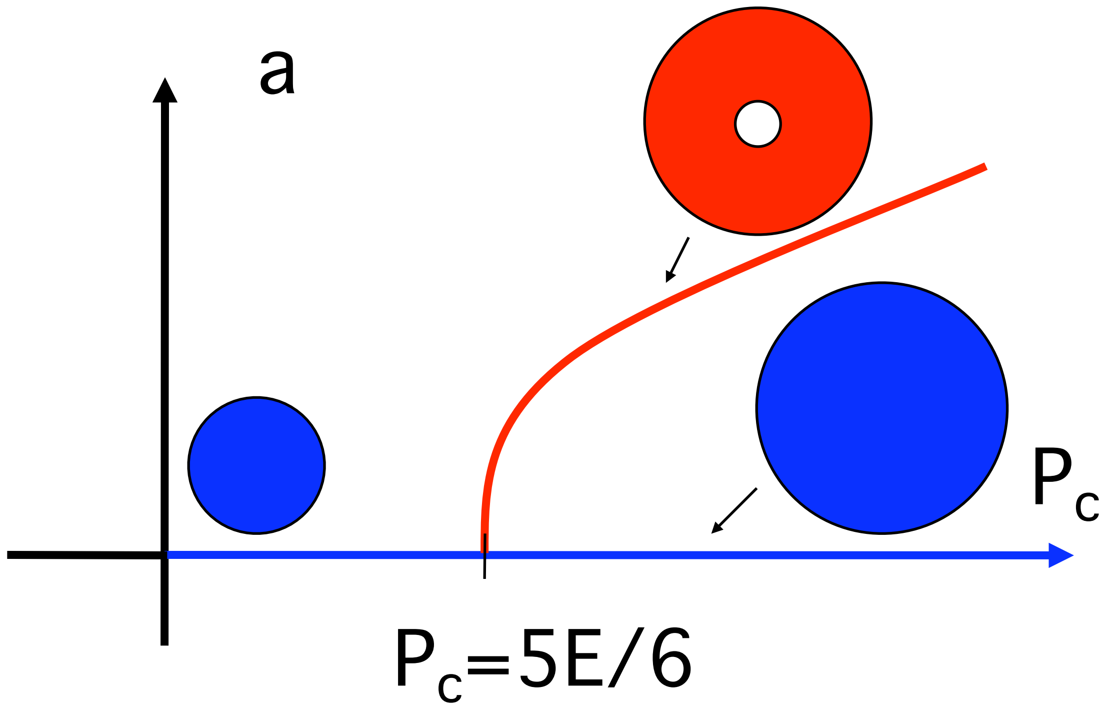
Bifurcation: Cavitation



Problems:

1. Topologically unstable
   $W=W_{nh}+\varepsilon\ \lambda^n$ with $n>2$
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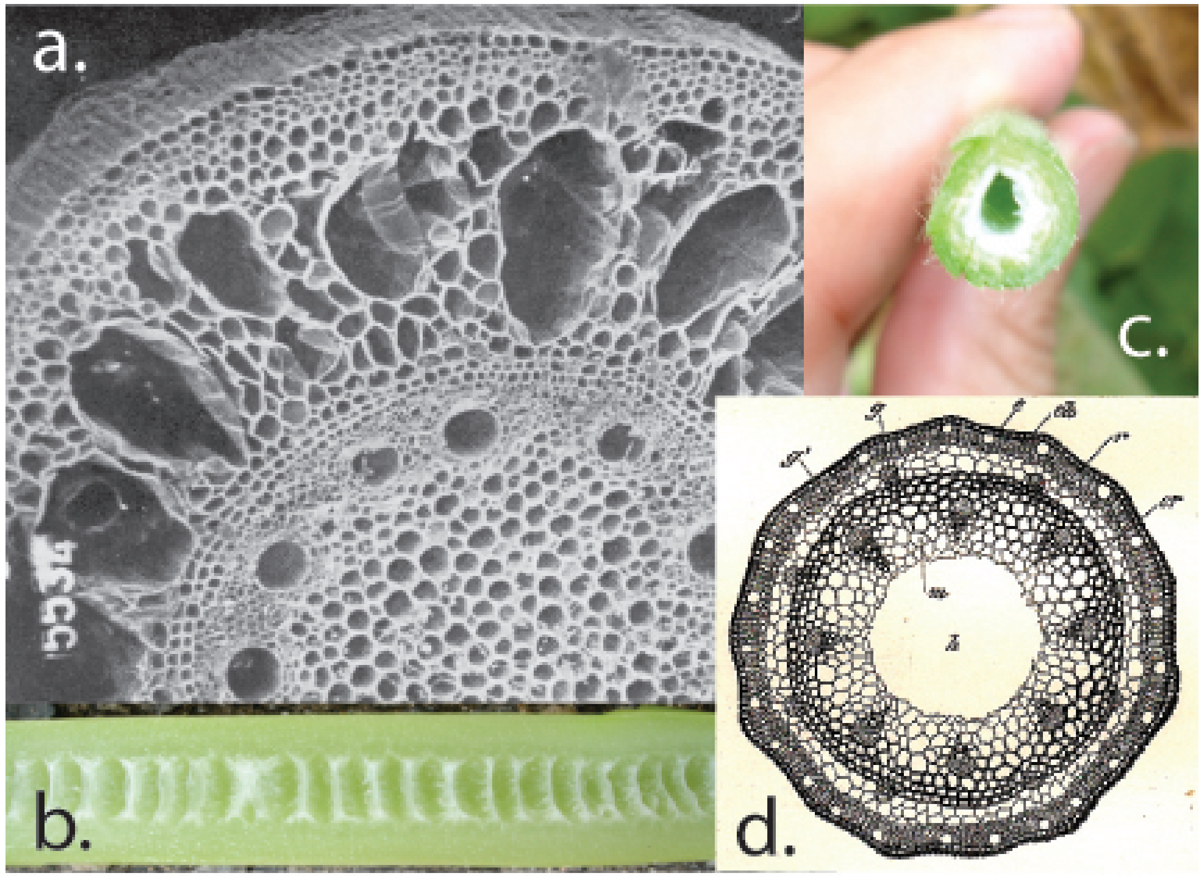
Schizengenous aerenchyma ➔

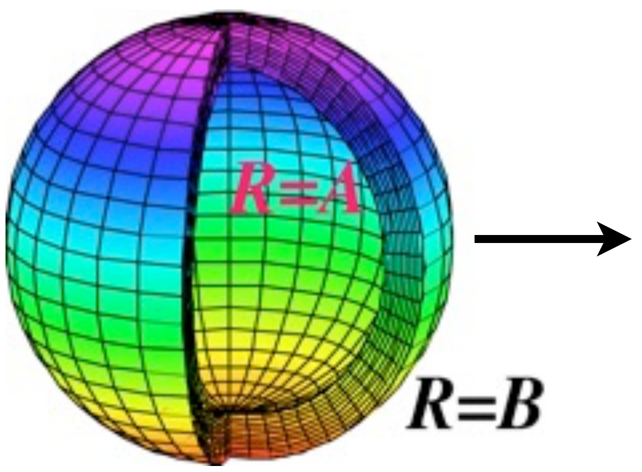


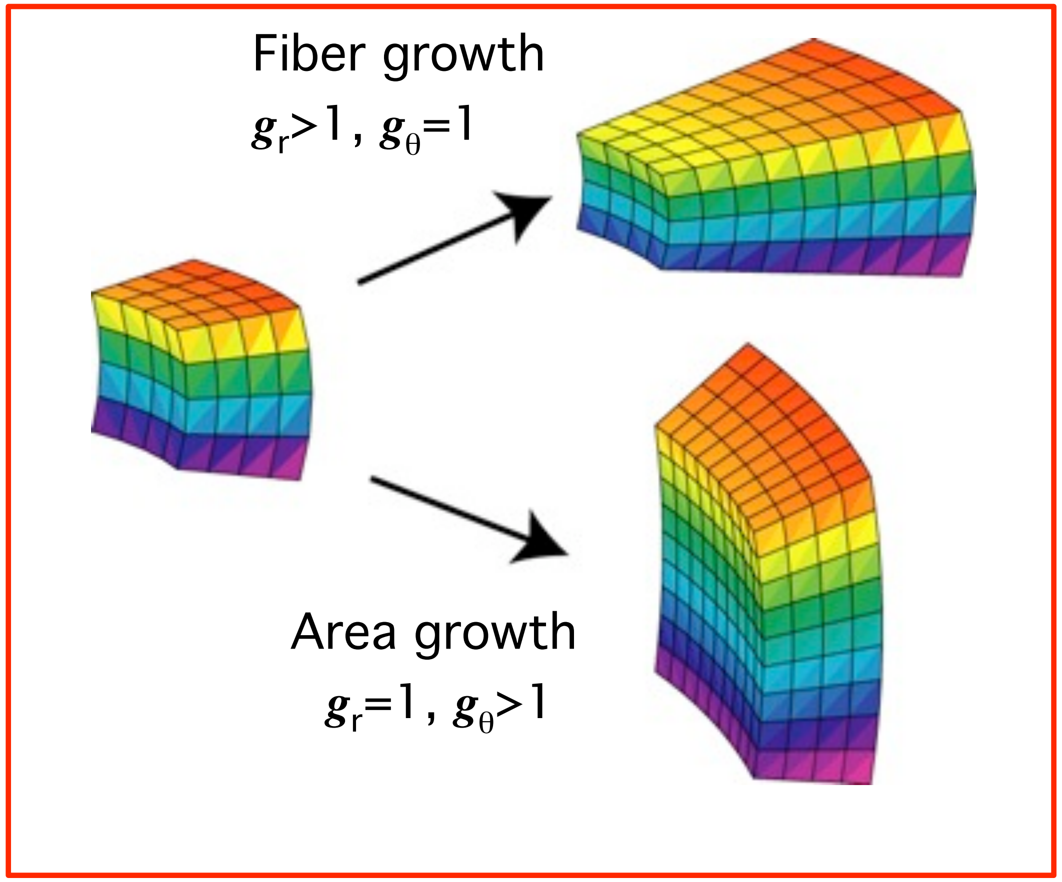
Ref: Ball ('82)

Ref: AG & Vandiver ('09)

Bifurcation: Cavitation

Consider the elastic growth of an elastically isotropic incompressible spherical shell





We have

$$\mathbf{F} = \begin{pmatrix} r' & 0 & 0 \\ 0 & r/R & 0 \\ 0 & 0 & r/R \end{pmatrix} \qquad \mathrm{G} = \begin{pmatrix} g_r & 0 & 0 \\ 0 & g_\theta & 0 \\ 0 & 0 & g_\theta \end{pmatrix} \qquad \mathrm{A} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

Bifurcation: Cavitation

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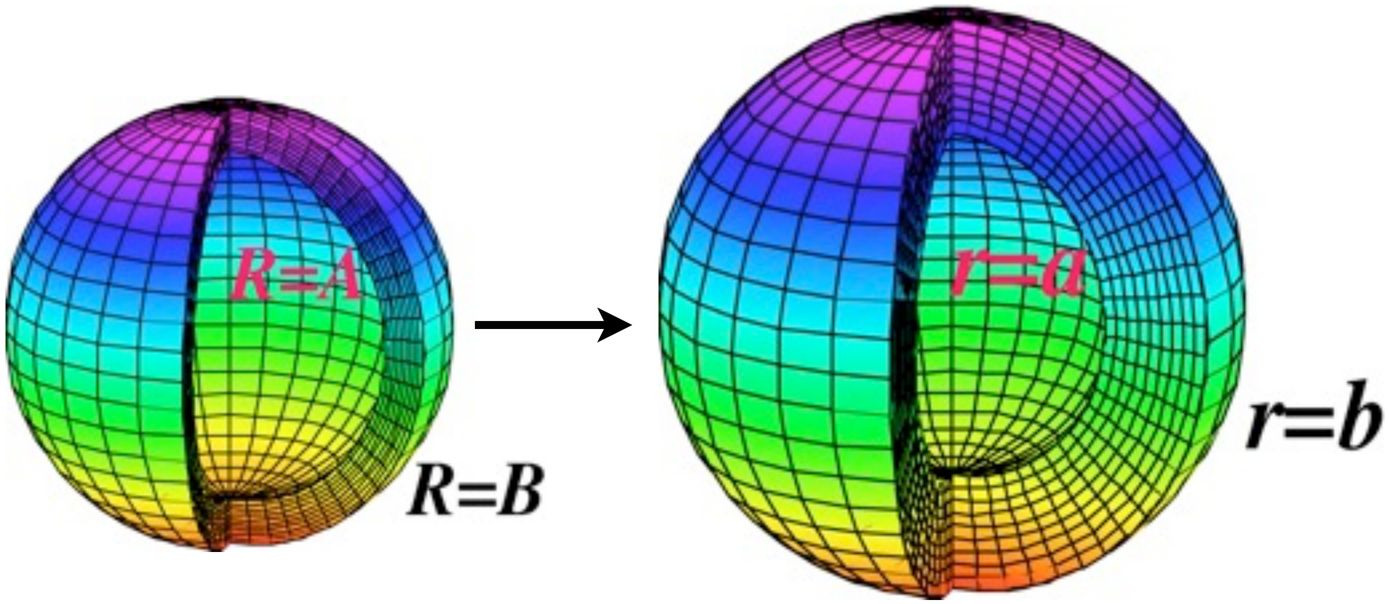

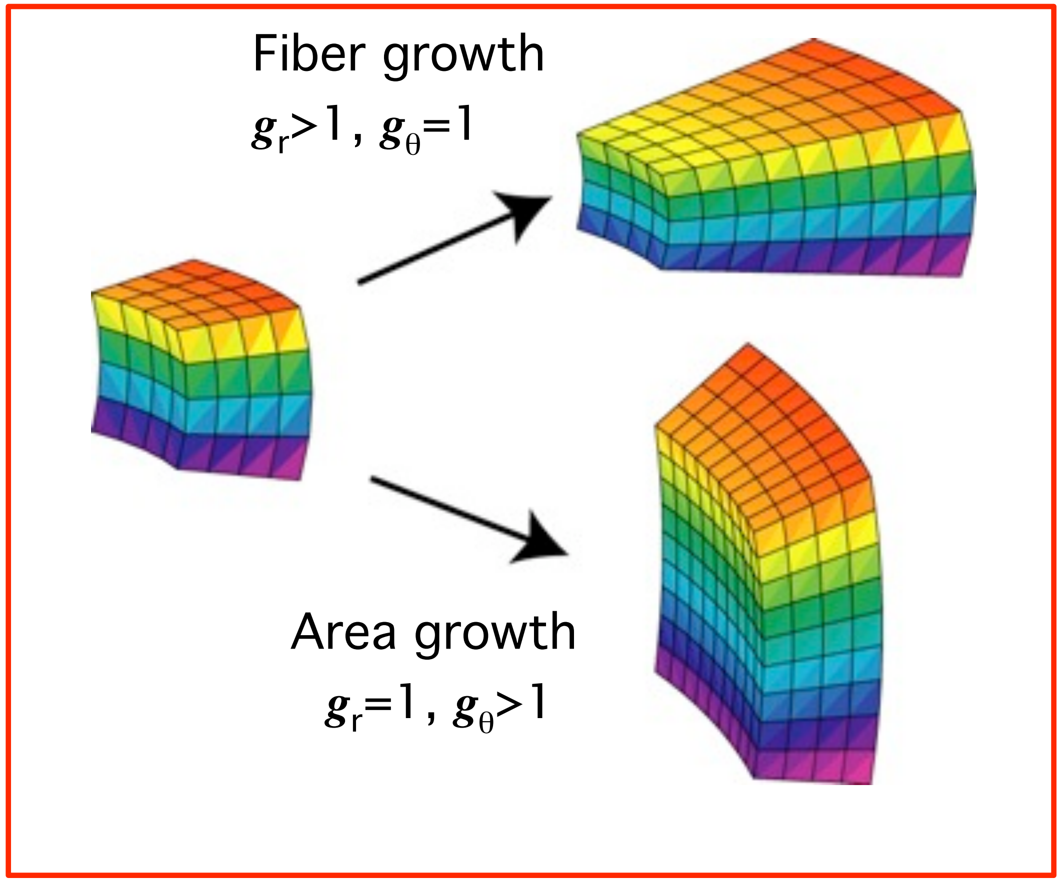

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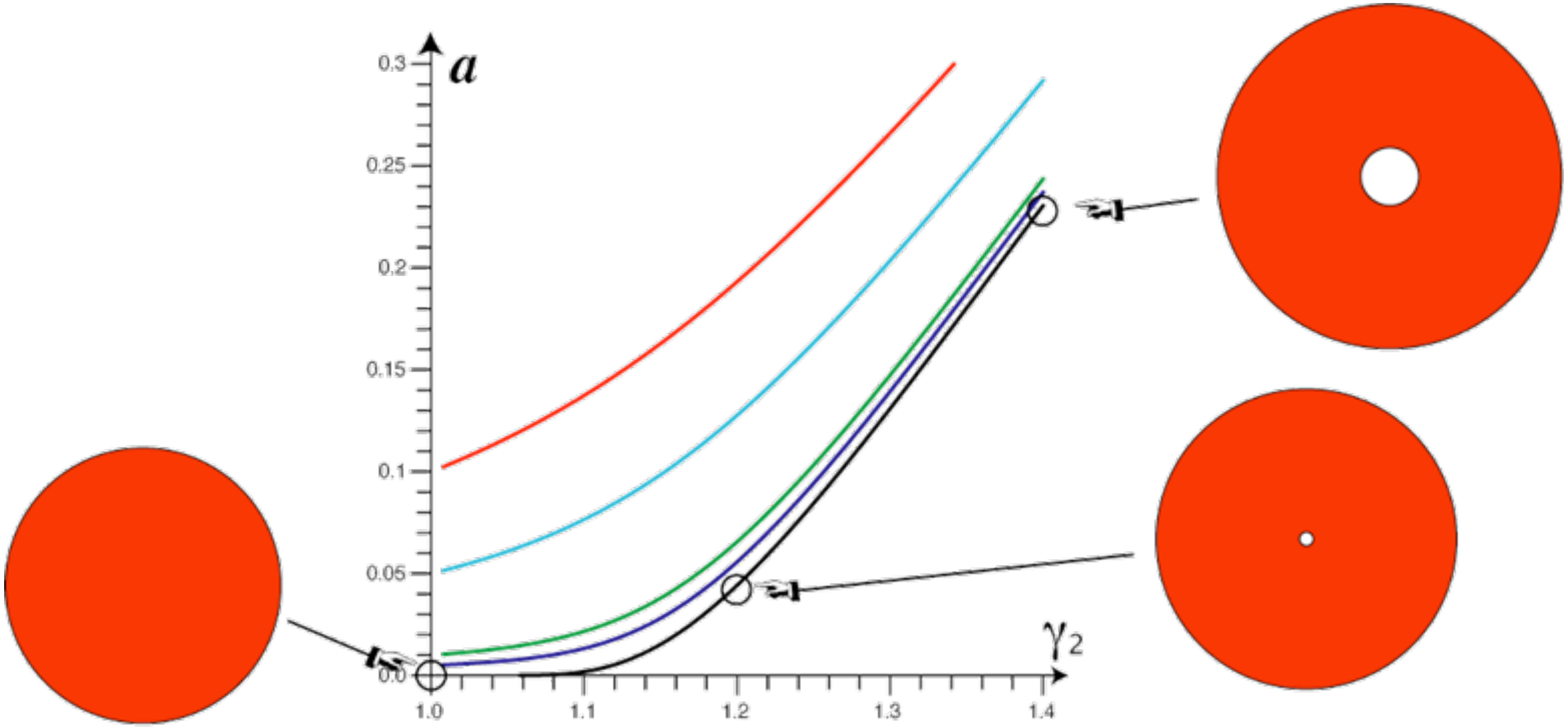
Bifurcation: Cavitation

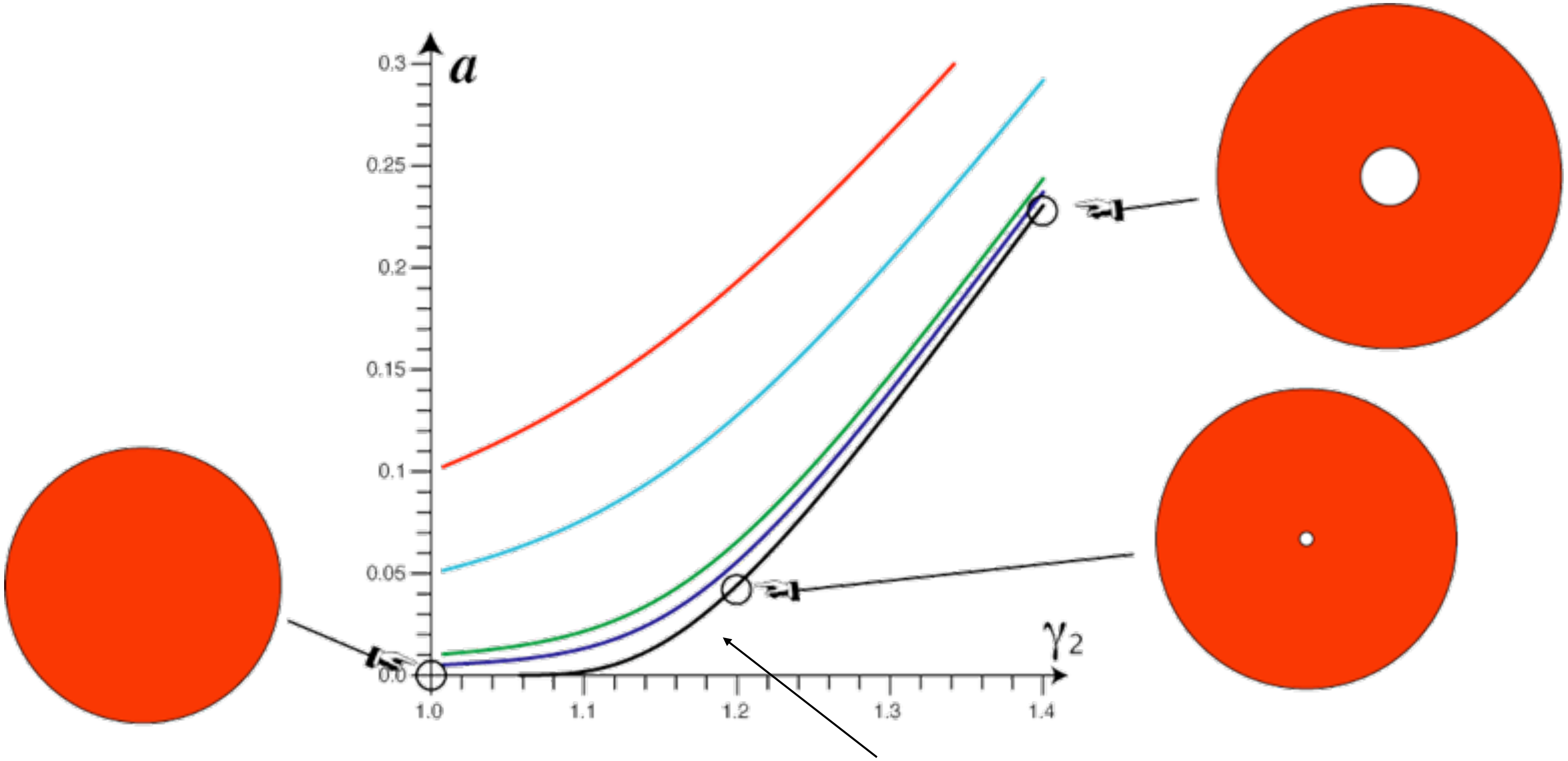
- Anisotropic growth
  - Area growth ($g_r=1,\ g_\theta>1$) creates tensile radial stress



Bifurcation: Cavitation

Anisotropic growth

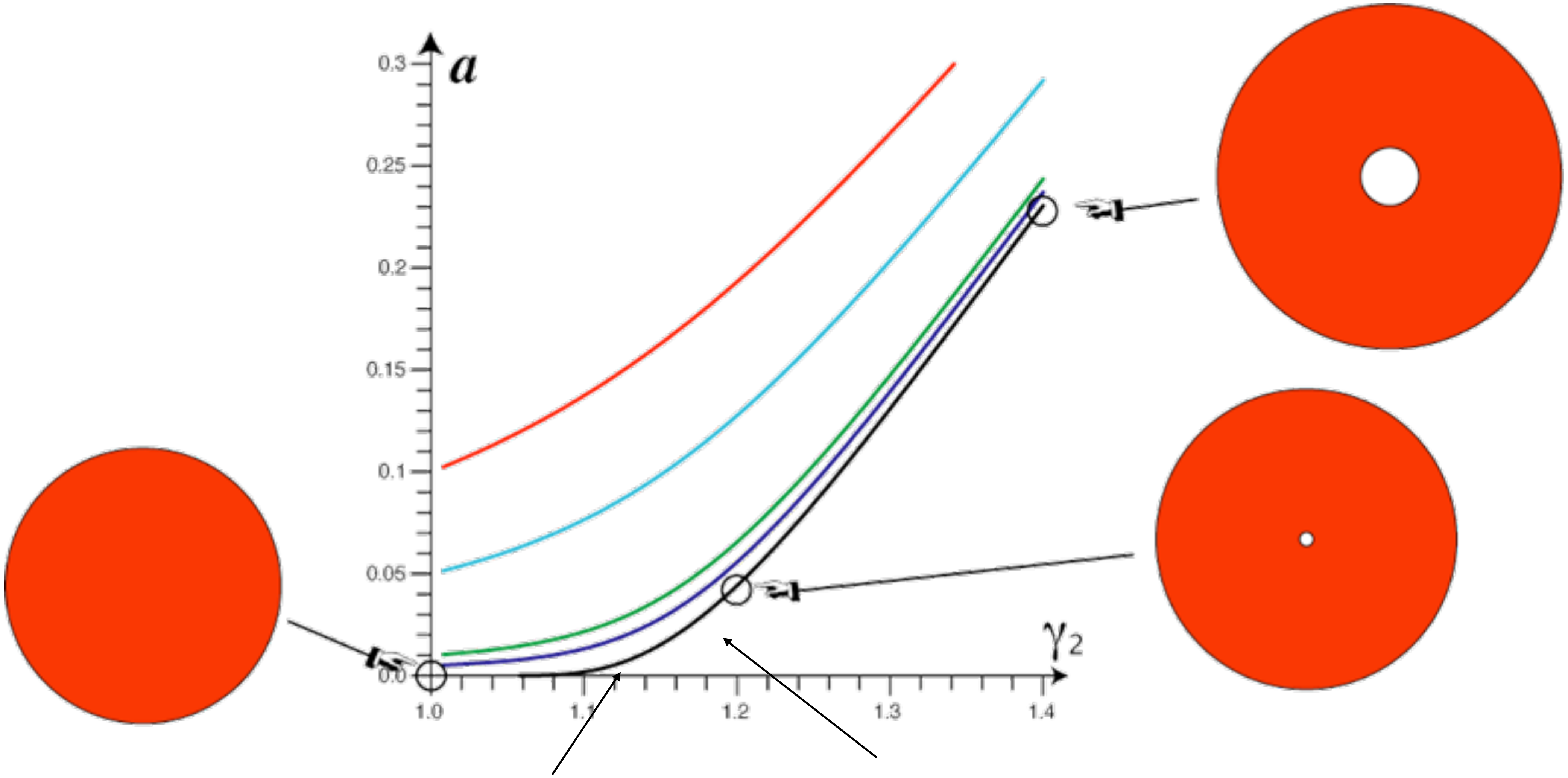
- Area growth ($g_r=1, g_\theta>1$) creates tensile radial stress



$$a \sim \sqrt{3}e^{\frac{\sqrt{3}\pi}{18}} \exp\left[-\frac{66g_\theta^2+183g_\theta-114}{36(5g_\theta+1)(g_\theta-1)}\right]$$

Bifurcation: Cavitation

- Anisotropic growth
  - Area growth ($g_r=1,\ g_\theta>1$) creates tensile radial stress



$$a \sim \sqrt{3}e^{\frac{\sqrt{3}\pi}{18}} \exp\left[-\frac{5/8}{g_\theta - 1}\right]$$

$$a \sim \sqrt{3}e^{\frac{\sqrt{3}\pi}{18}} \exp\left[-\frac{66g_\theta^2 + 183g_\theta - 114}{36(5g_\theta + 1)(g_\theta - 1)}\right]$$

Cavitation occurs for arbitrarily small values of g_θ

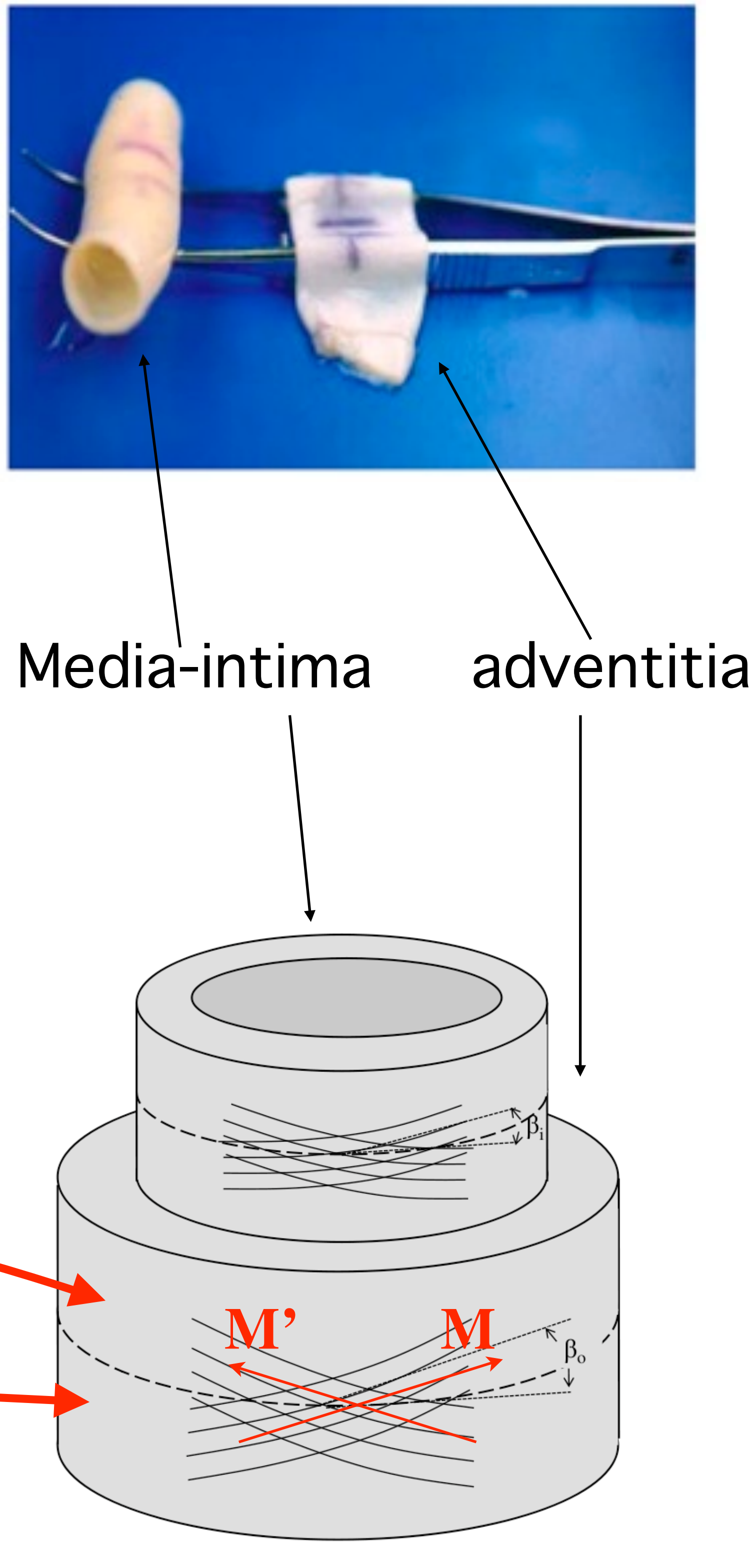
Instability: arteries

- Model

  Double cylinder with residual stress
  under loads and pressure

- Elastic model (Holzapfel)
  - Two layers: NeoHokean with stiffening (Fung) fibers

$$W = \frac{\mu}{2}(I_1 - 3) + \frac{k_1}{2\nu} \sum_{n=4,6} \left(e^{\nu (I_n - 1)^2} - 1 \right)$$

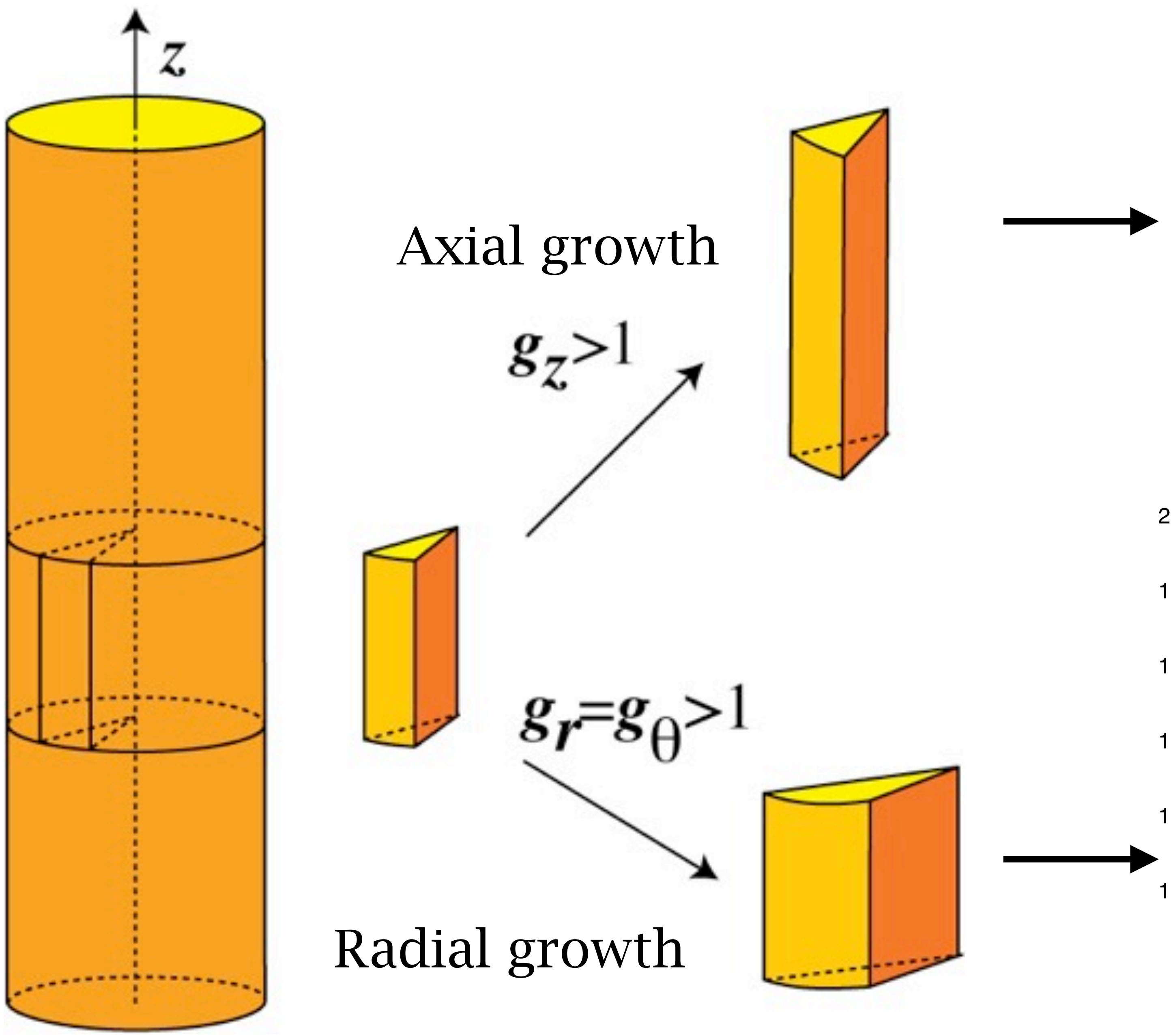


$$I_1 = \mathrm{tr}(\mathbf{C})$$
$$I_4 = \mathbf{C} : \mathbf{M} \otimes \mathbf{M}$$
$$I_6 = \mathbf{C} : \mathbf{M}' \otimes \mathbf{M}'$$

$$\mathbf{C} = \mathbf{A}^T \cdot \mathbf{A}$$

Right Cauchy-Green tensor

Instability: arteries

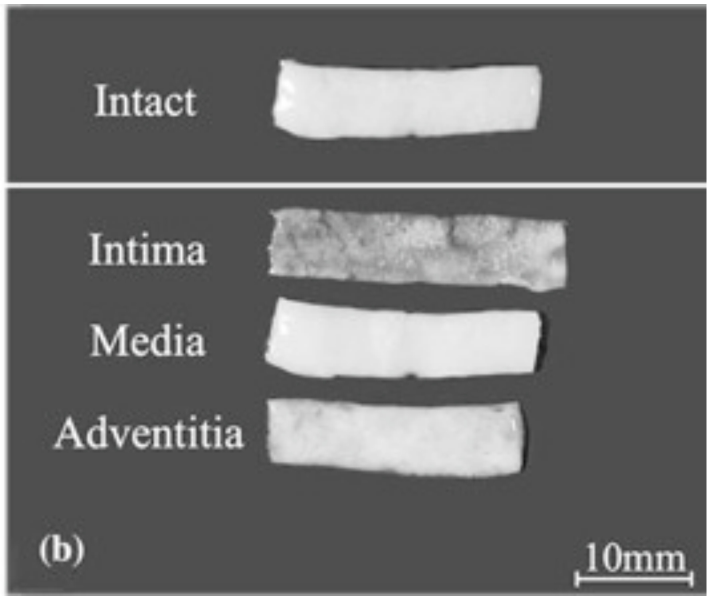


Axial growth

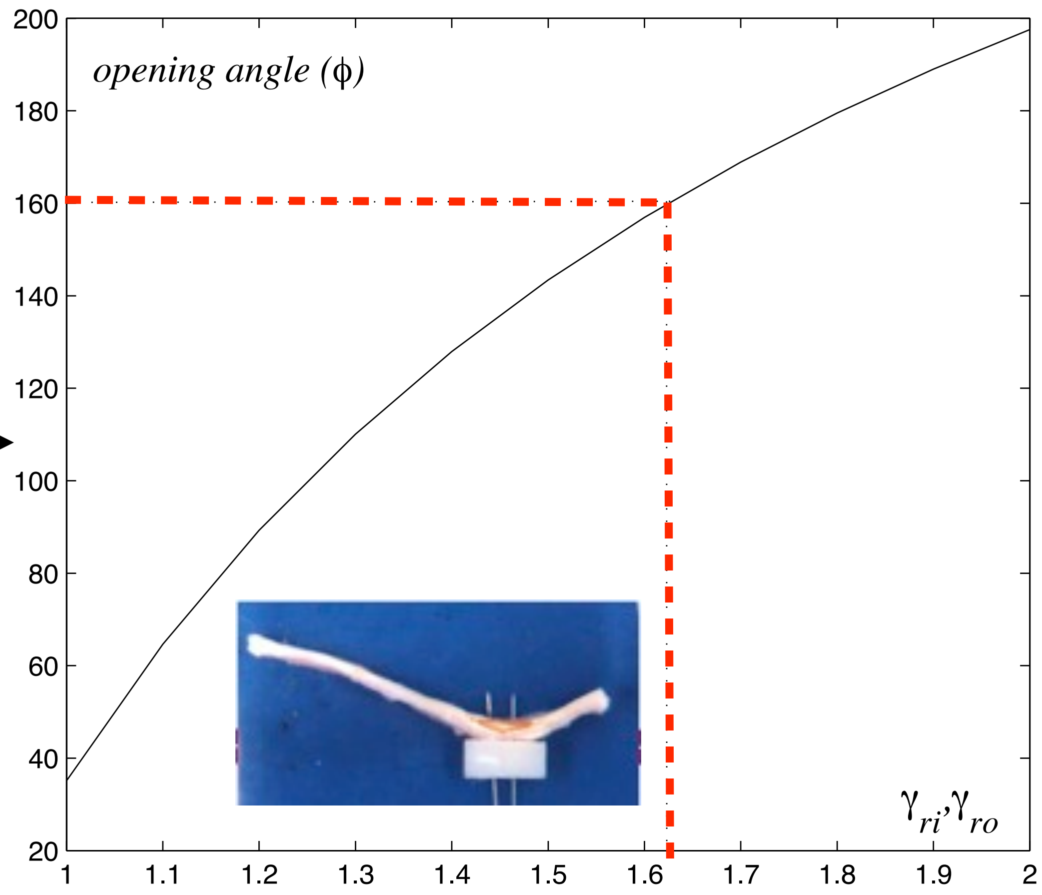
$g_z > 1$

$g_r = g_\theta > 1$

Radial growth



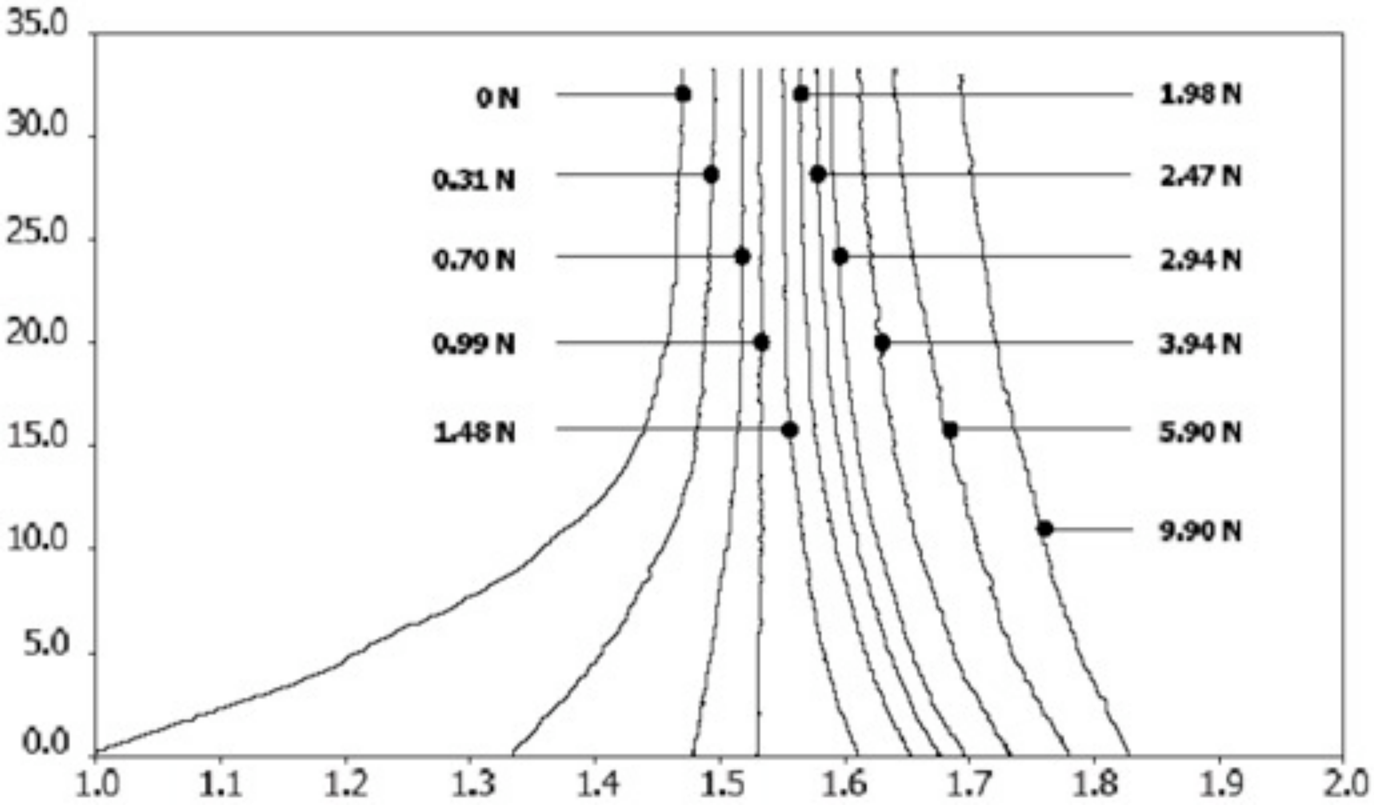
Holzapfel et al. ’07



Instability: arteries

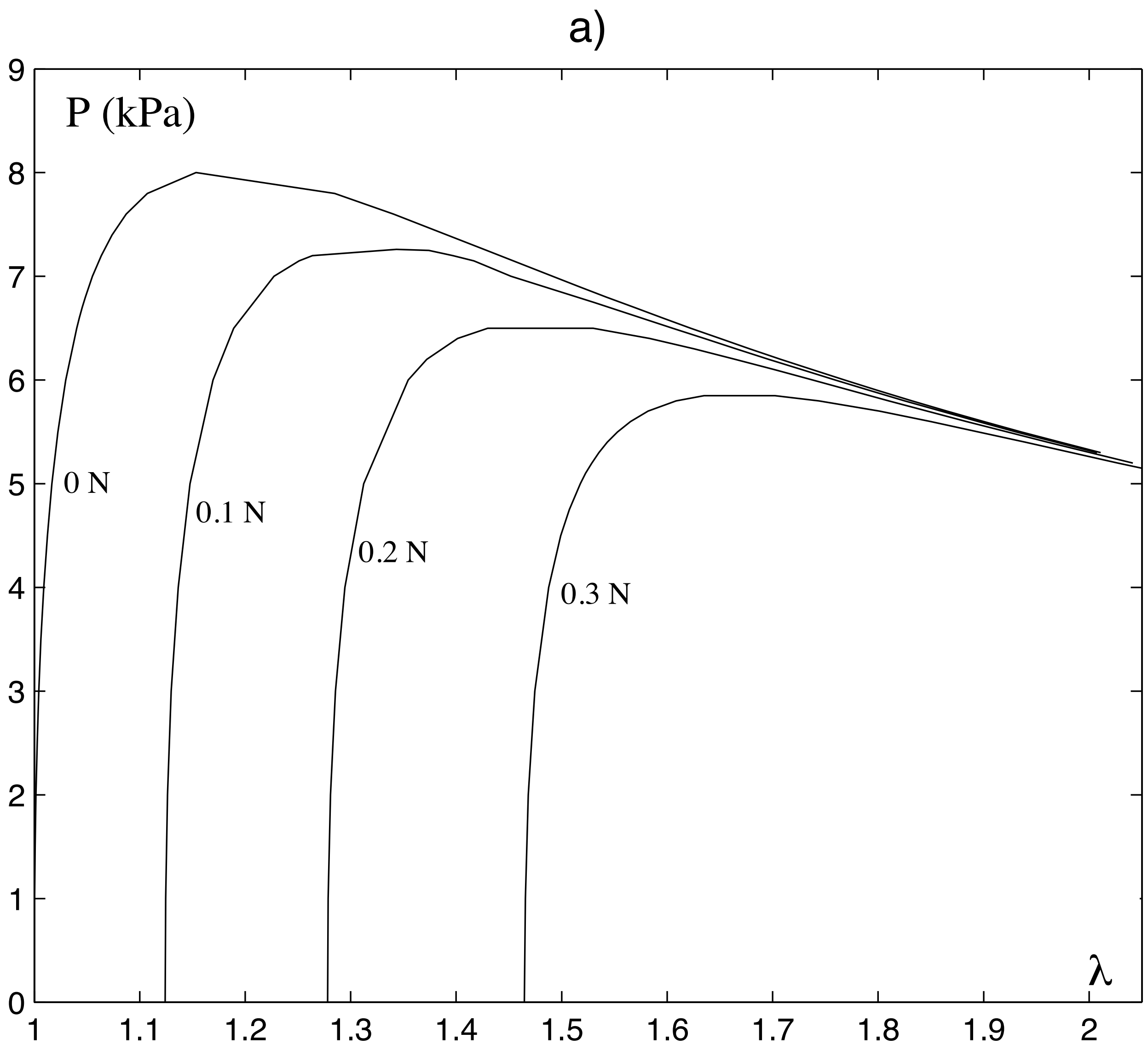
- Experimental curves

Axial stretch for fixed tension,
varying pressure



Holzapfel-Ogden

- Neo-Hookean model of arteries



$$\boxed{W = \frac{\mu}{2}(I_1 - 3) + \frac{k_1}{2\nu}\xcancel{\sum_{n=4,6}\left(e^{\nu(I_n-1)^2} - 1\right)}}$$

Instability: arteries

- Experimental curves
- Fung model of arteries

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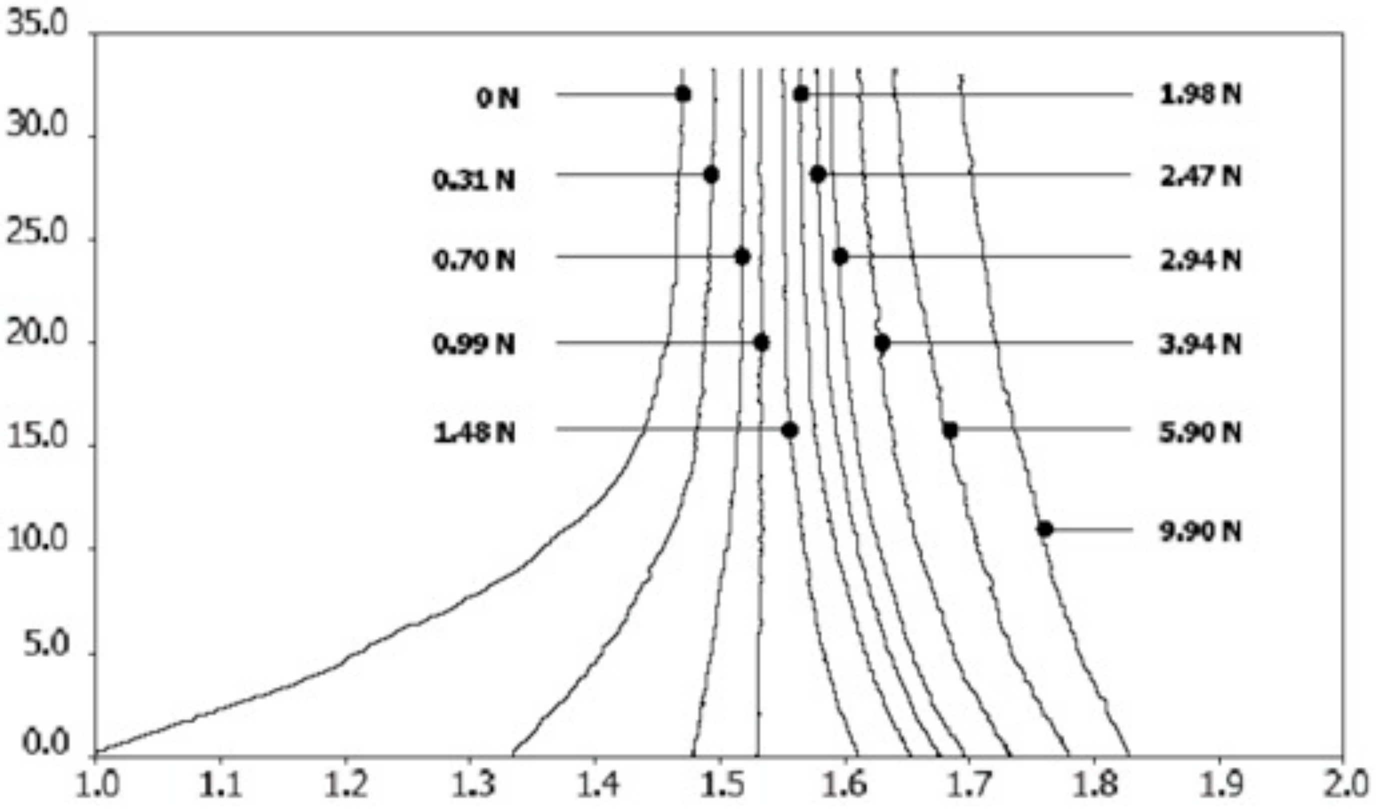
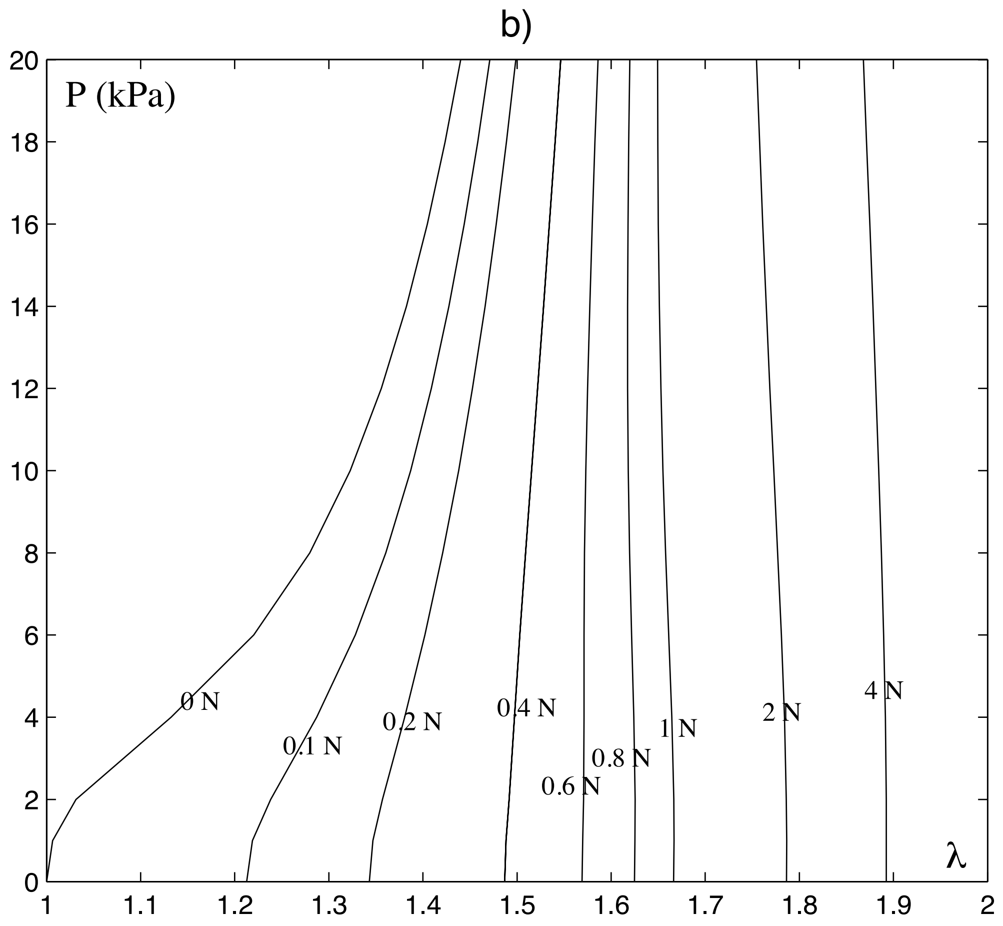


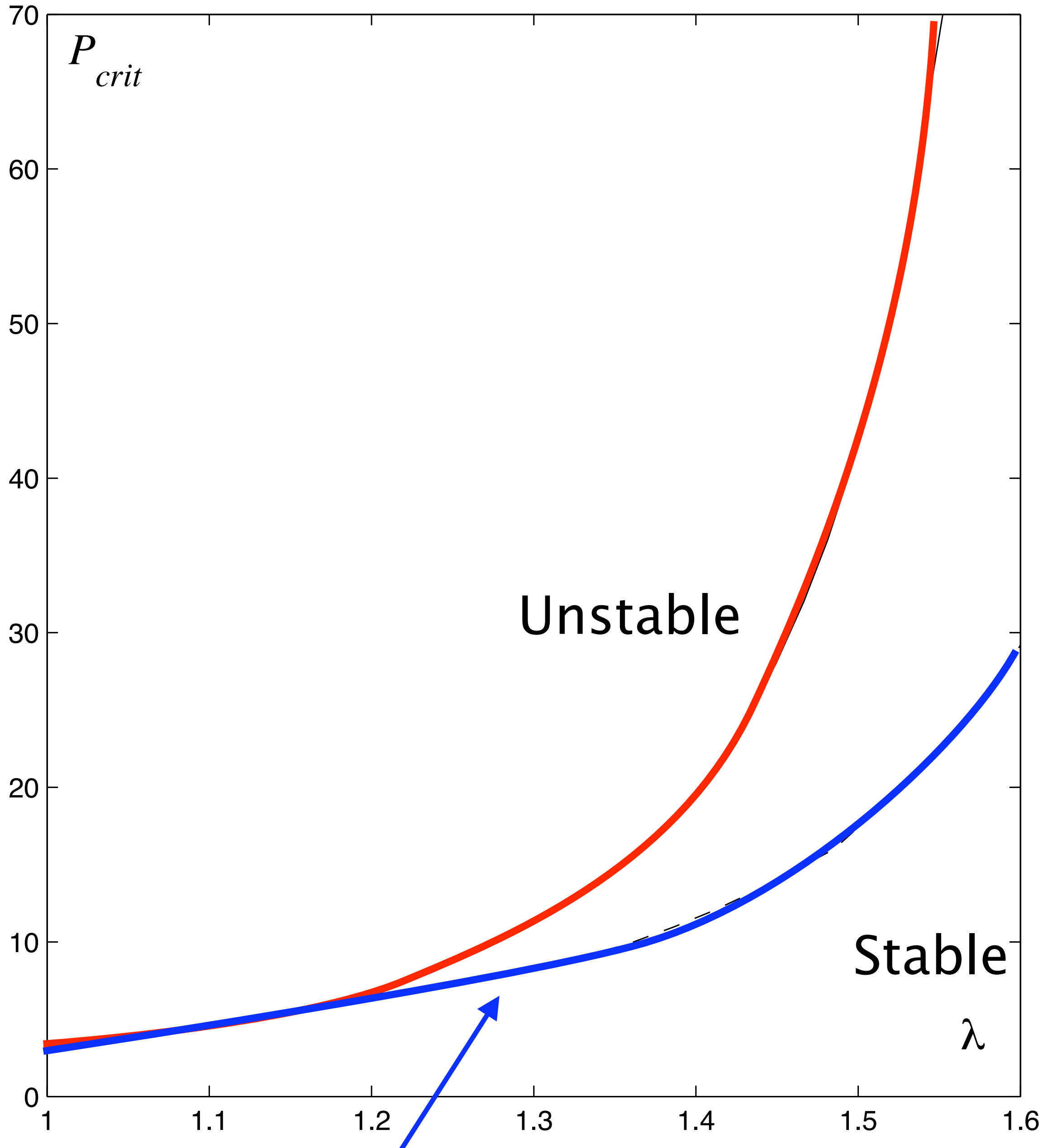


Holzapfel-Ogden

$$W = \frac{\mu}{2}(I_1 - 3) + \frac{k_1}{2\nu}\sum_{n=4,6}\left(e^{\nu(I_n-1)^2} - 1\right)$$

Instability: arteries

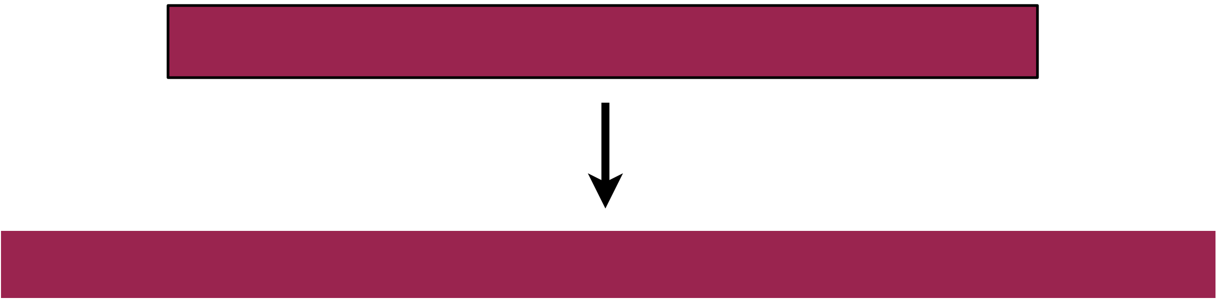
- Instability

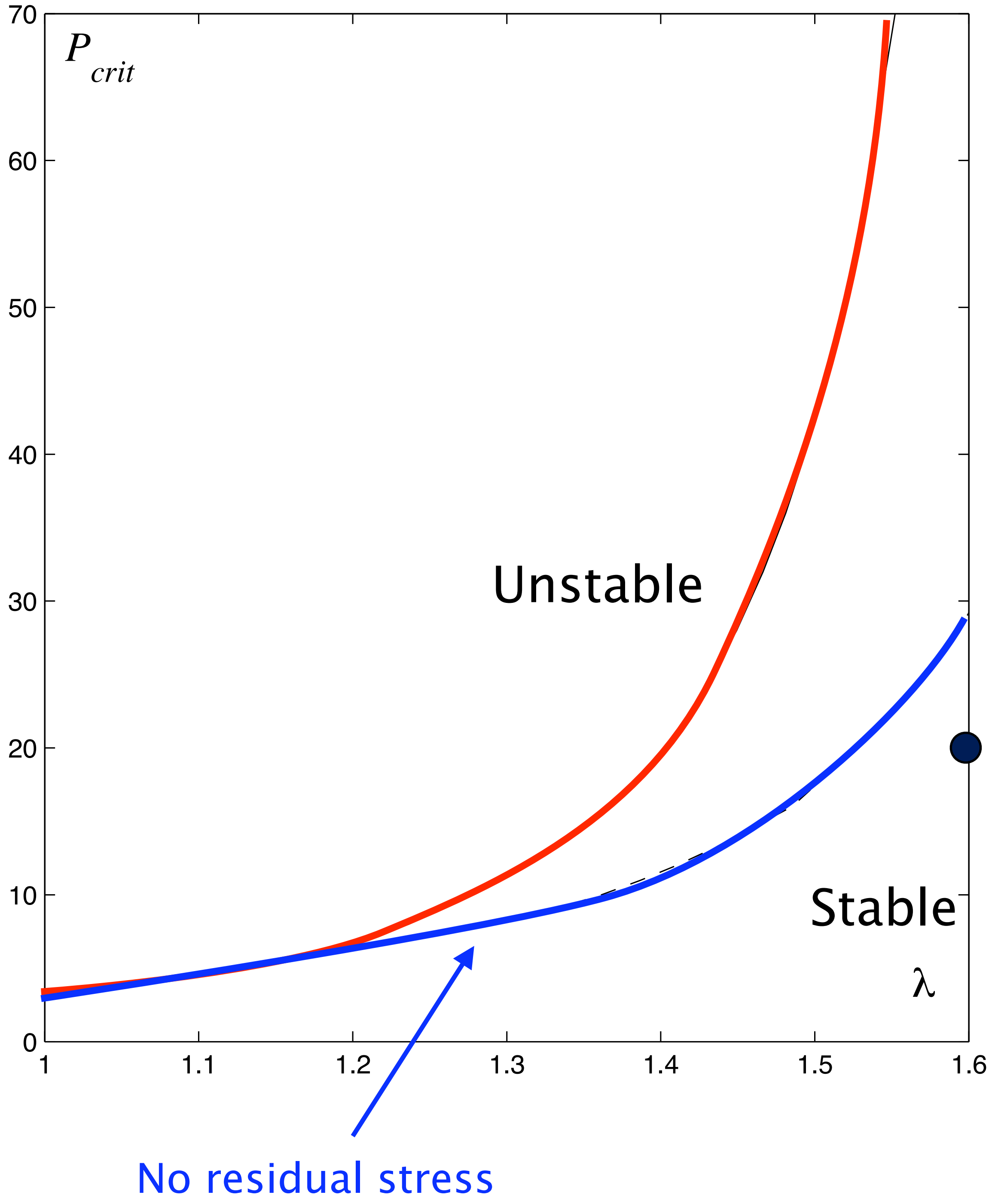


No residual stress

Instability: arteries

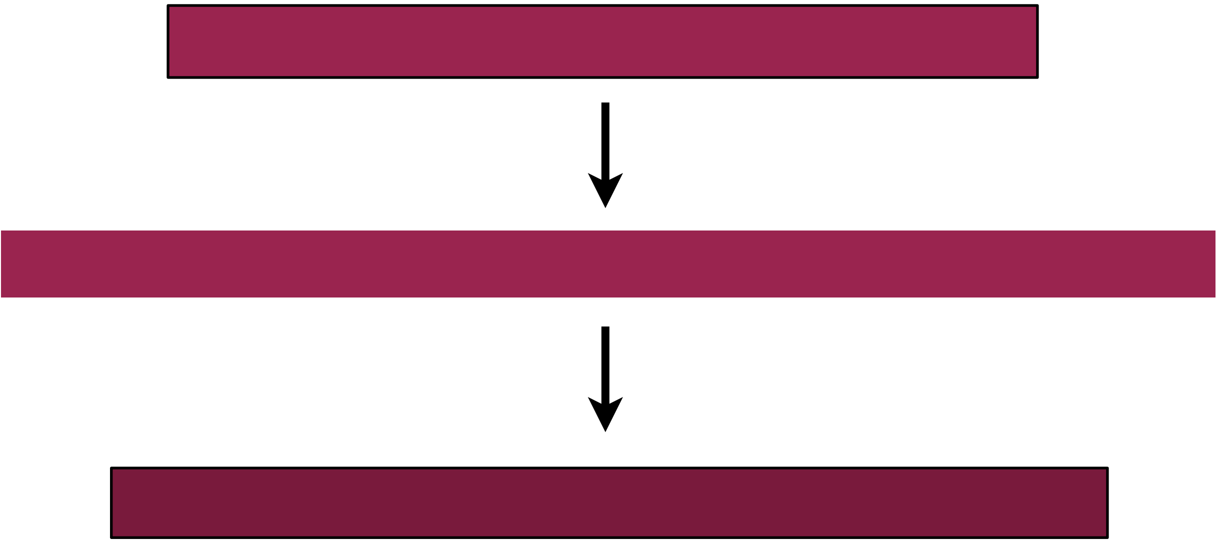
- Instability

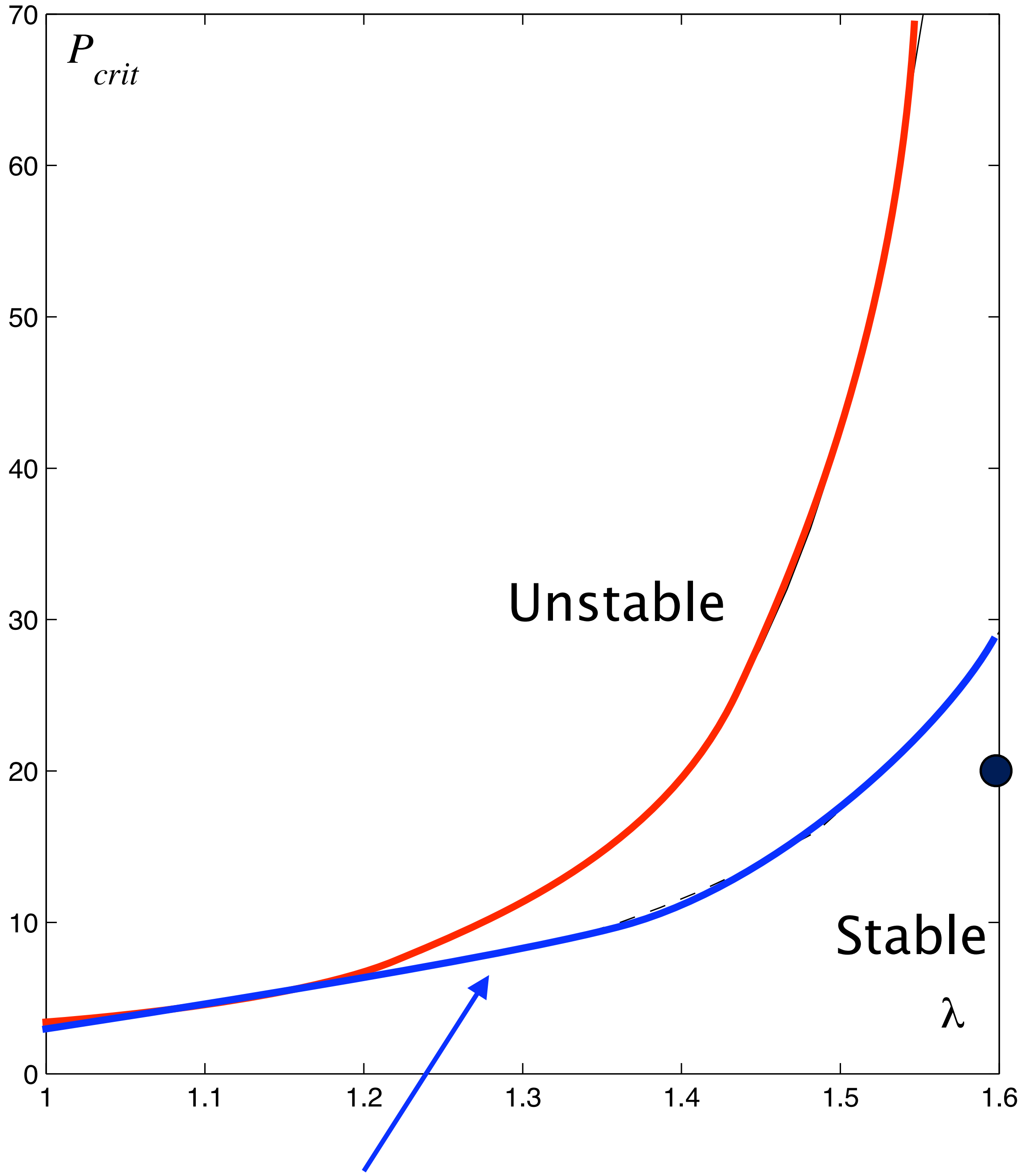




Instability: arteries

- Instability

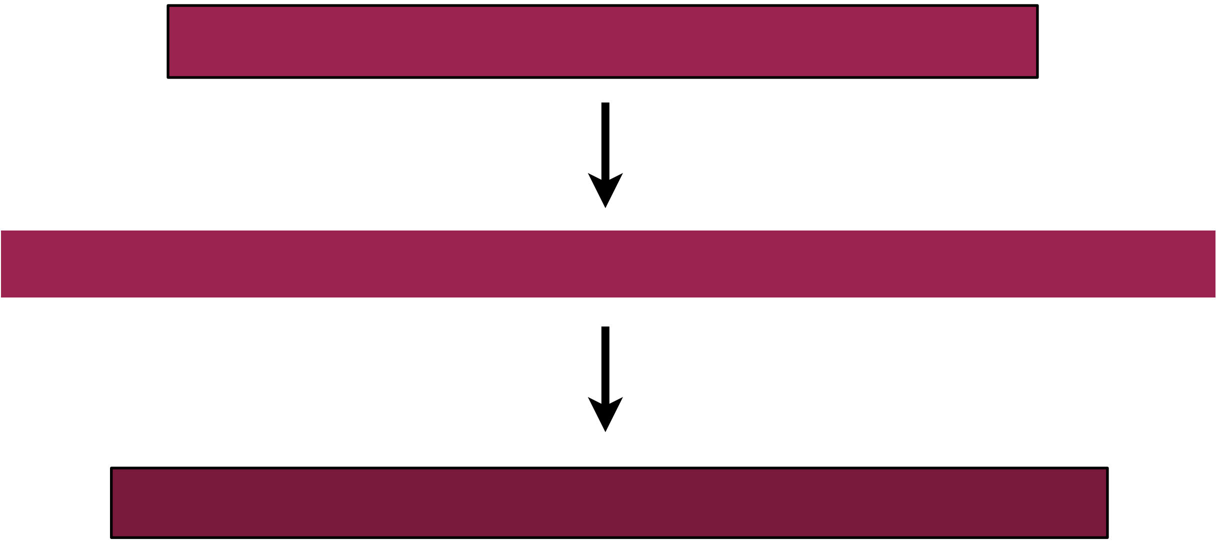


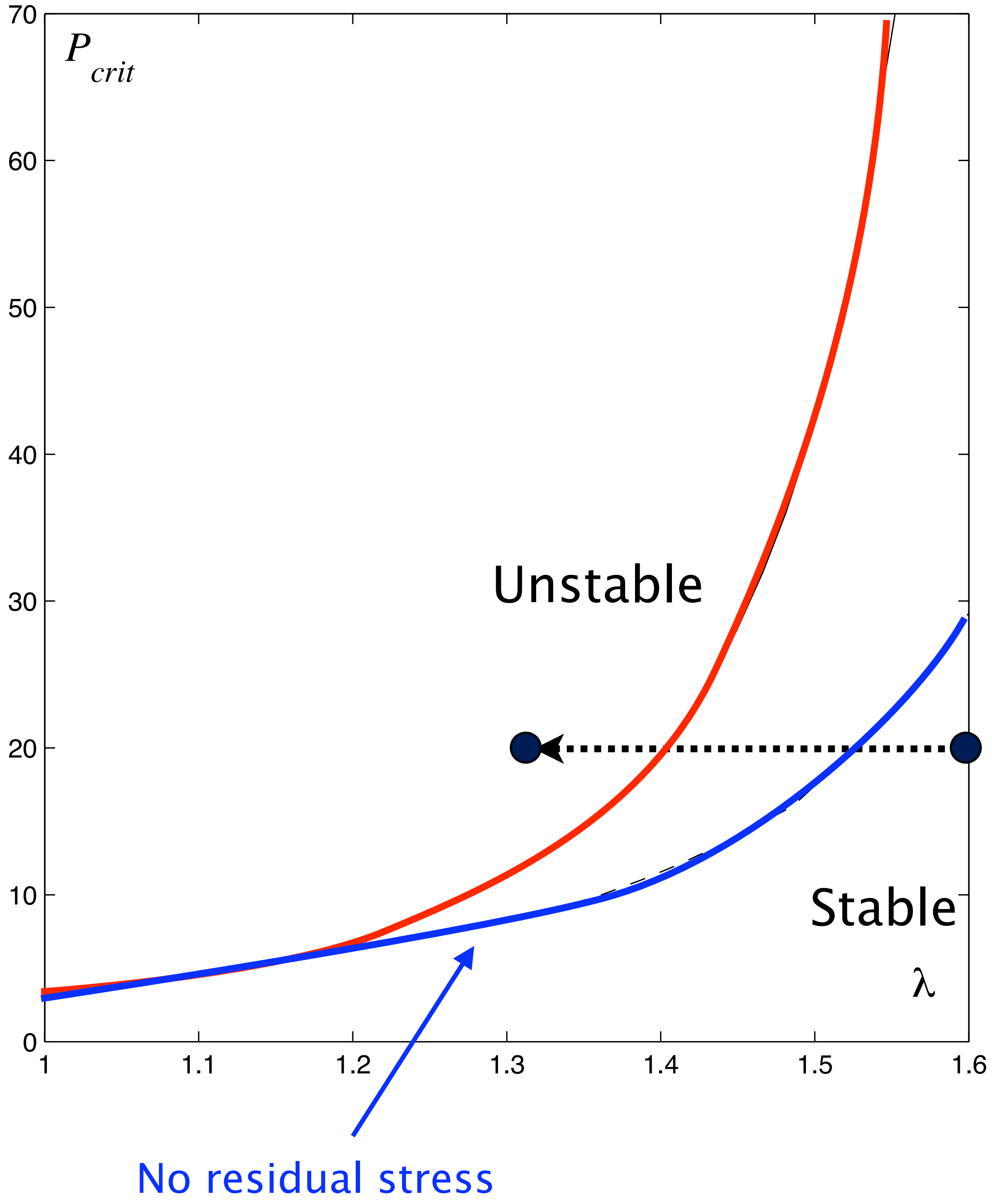


No residual stress

Instability: arteries

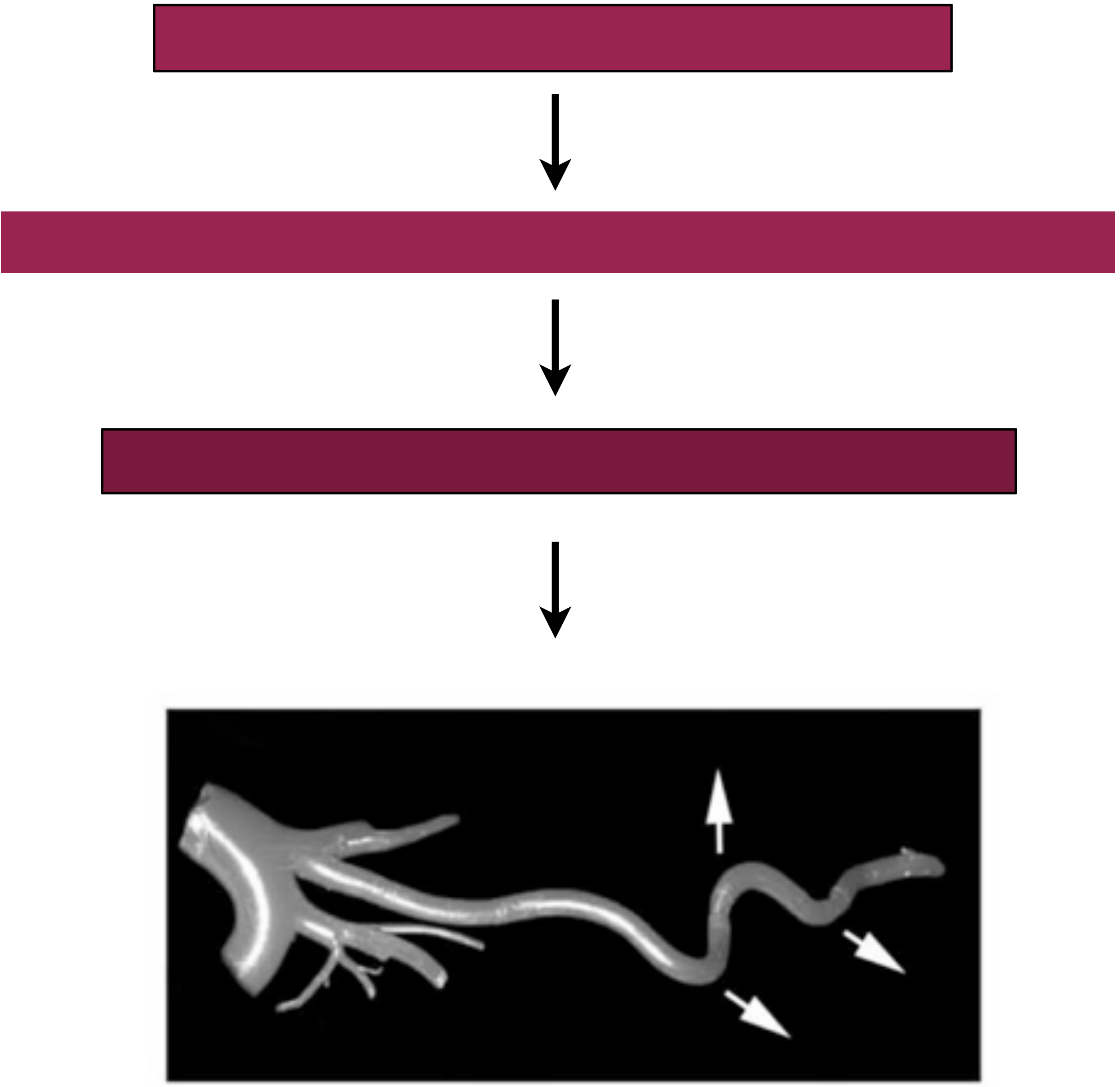
- Instability



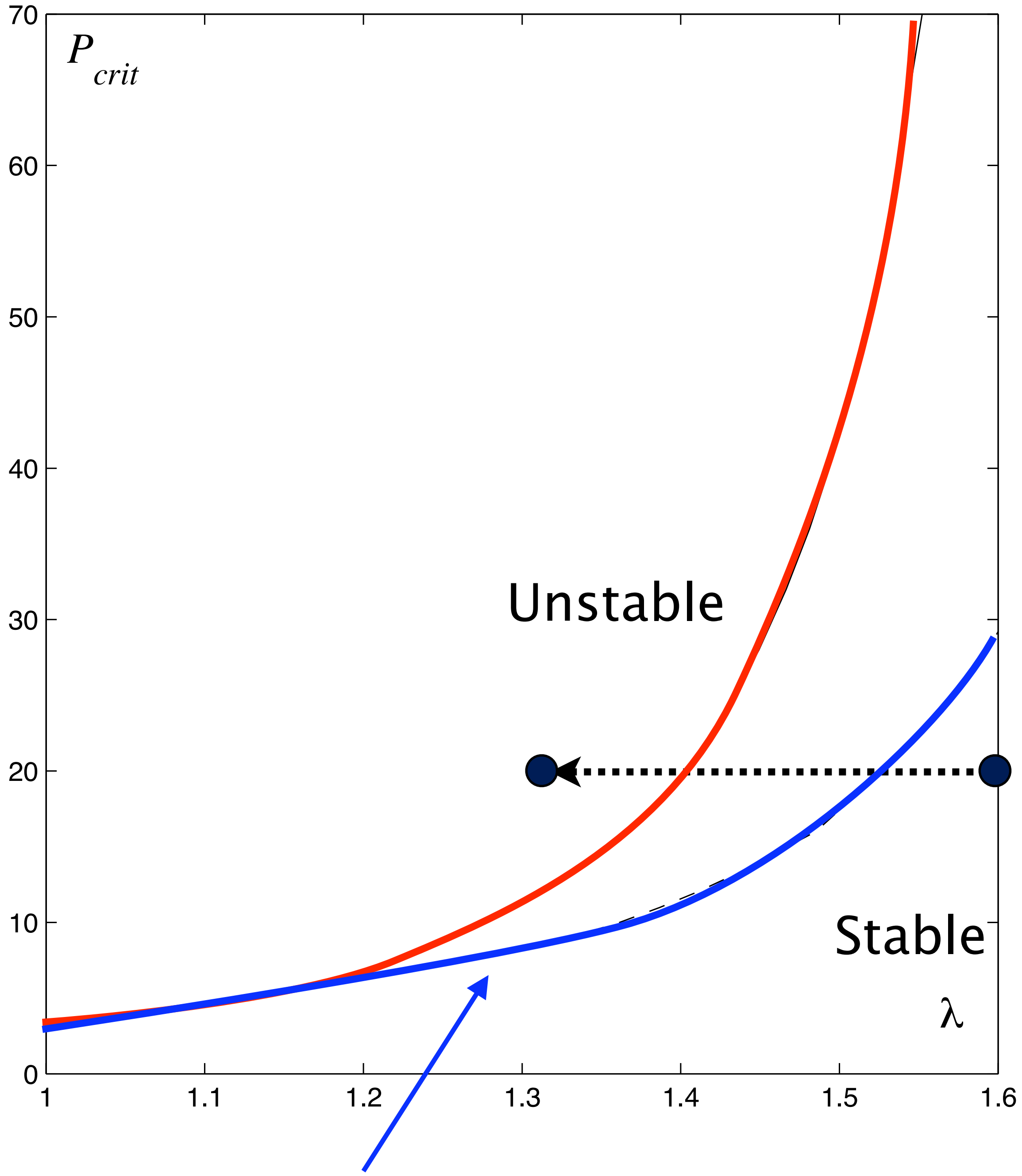


Instability: arteries

- Instability



Jackson et al. 2005



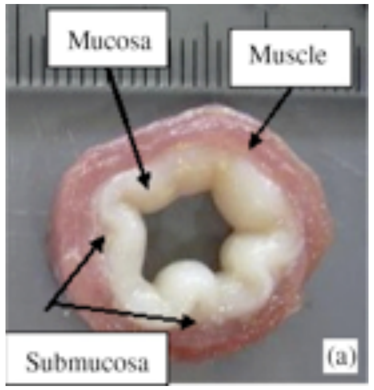
No residual stress

Ref: Vandiver & AG (preprint ’09)

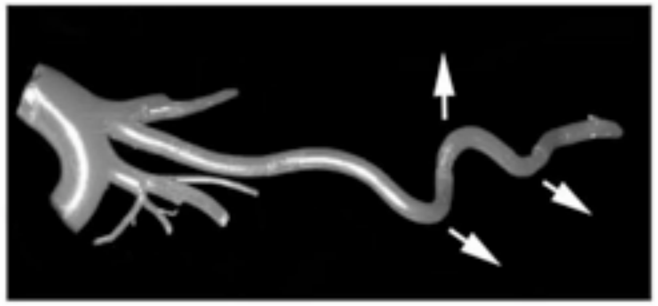
Summary & Perspectives

1. Morpho-elasticity

- Stress ⇔ Growth
- Nonlinear elastic response
- Anelastic response
- Conceptual framework
- Computational framework



mucosal folding,
Yang et al. (‘07)



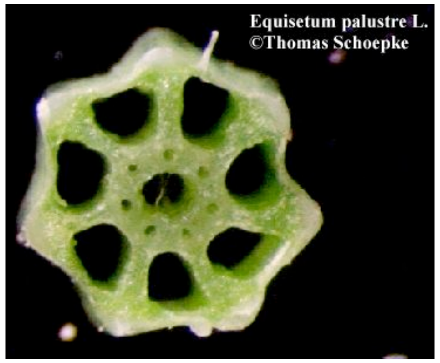
Buckling of arteries Jackson (’05),

2. Methods

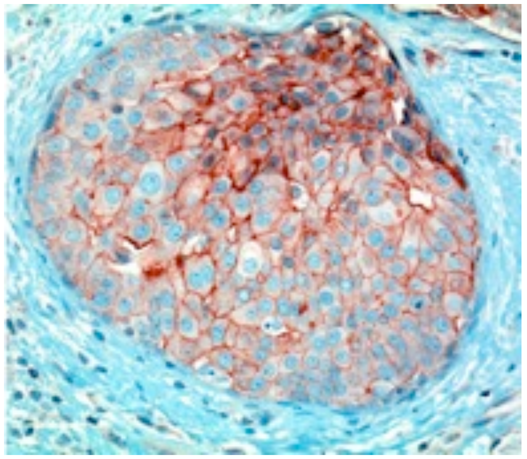
- Long-time dynamics
- Bifurcation & Stability
- Pattern formation

3. Theory

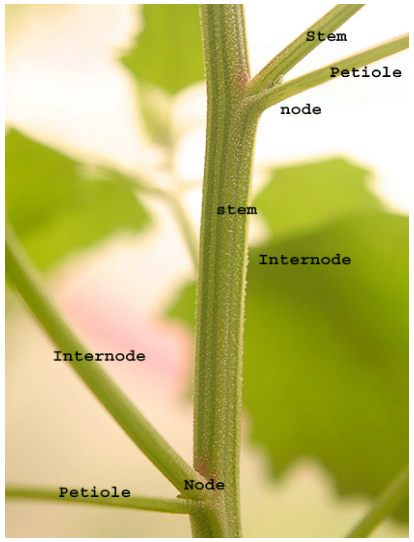
- Differential geometry
- Growth laws
- Coupling (diffusion,...)
- 1D morphoelasticity
- 2D morphoelasticity



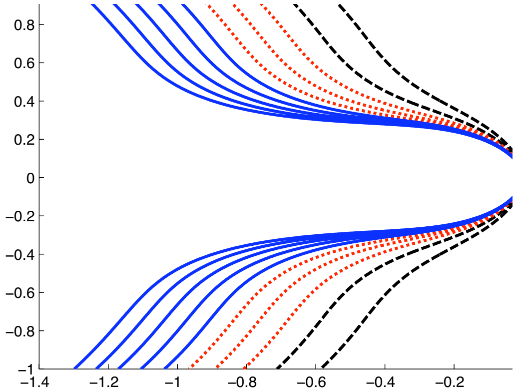
Stem hollowing
Vandiver & AG (09)



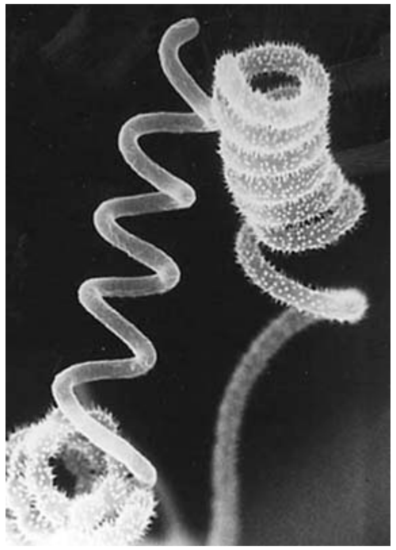
Solid tumors (Gatenby)



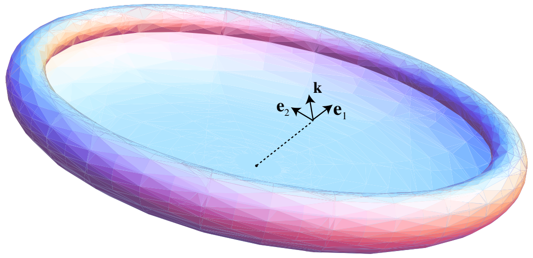
Stem rigidity
Vandiver & AG (08)



Tip growth
Tabor & AG



Helical growth



Growing plates
Mc Mahon, Tabor & AG