Mechanisms of Instability in Nearly Integrable Hamiltonian Systems

A. Delshams, R. de la LLave,T. M. Seara

papers: Comm. Math. Phys. 2000, ERA Amer. Math. Soc. 2003, Mem. Amer. Math. Soc. 2006, Adv. Math. 2006, Adv. Math. 2008.

M. Gidea, G. Huguet, V. Kaloshin

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Outline of the talk

- Instabilities are also called diffusion; what is diffusion?
- Heuristics about the method
- Arnol'd example
- Geodesic flows: diffusion with small gaps:
- Diffusion in the (elliptic) restricted planar three body problem: no gaps but exponentially small phenomena.
- Diffusion with big gaps: a priori unstable systems
- Diffusion in higher dimensions

Main question:

Understand how small forces produce large effects in mechanical systems without friction

What is diffusion?

Diffusion $\equiv$ Gaining lots of energy by applying small forces.

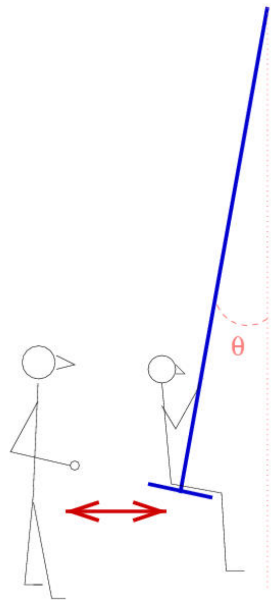
Diffusion $\equiv$ changes of order 1 in the actions (instabilities) for arbitrarily small perturbations of integrable systems.

If you apply a periodic perturbation to a system. Will the perturbation accumulate or will it average out?

Example: forced pendulum

$$x'' + \sin x = \varepsilon F(t)$$

$$H(y, x, t; \varepsilon) = \frac{y^2}{2} - \cos x + \varepsilon \varphi F(t),$$



1. A harmonic oscillator gains energy when (and only when) is pushed with the same frequency as its natural frequency.

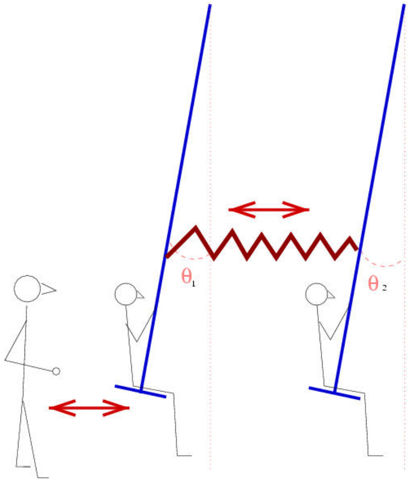
$$\varphi'' + \varphi = \cos t \qquad \Longrightarrow \varphi(t) = \frac{1}{2} t \sin(t) + A \sin(t) + B \cos(t)$$

2. If the swing has more energy its frequency changes.

So, if the swing gains energy, it moves to an area when it does not gain energy.

This previous argument was made rigorous in the late 50's in the work of Kolmogorov Arnol'd and Moser for small forcing.

The situation is different for periodic external perturbations of systems of two or more degrees of freedom.



There exist orbits whose energy increases slowly, but significantly.

What is diffusion?

$$H(I,\varphi,t;\varepsilon) = H_0(I) + \varepsilon H_1(I;\varphi,t;\varepsilon),$$

where

$$(I,\varphi,t) \in \mathbb{R}^d \times \mathbb{T}^{d+1}$$

when $\varepsilon = 0$, the motion is

$$I = I_0; \quad \varphi = \varphi_0 + \nabla H_0(I_0)t$$

All the motion is confined in $d+1$-dimensional tori:

$$\mathbb{T}_I = \{(I,\varphi,t),\ (\varphi,t) \in \mathbb{T}^{d+1}\}$$

and the motion in $\mathbb{T}_I$

is quasi-periodic with frequency $\omega(I) = (\nabla H_0(I), 1)$

what happens when $\varepsilon \neq 0$?

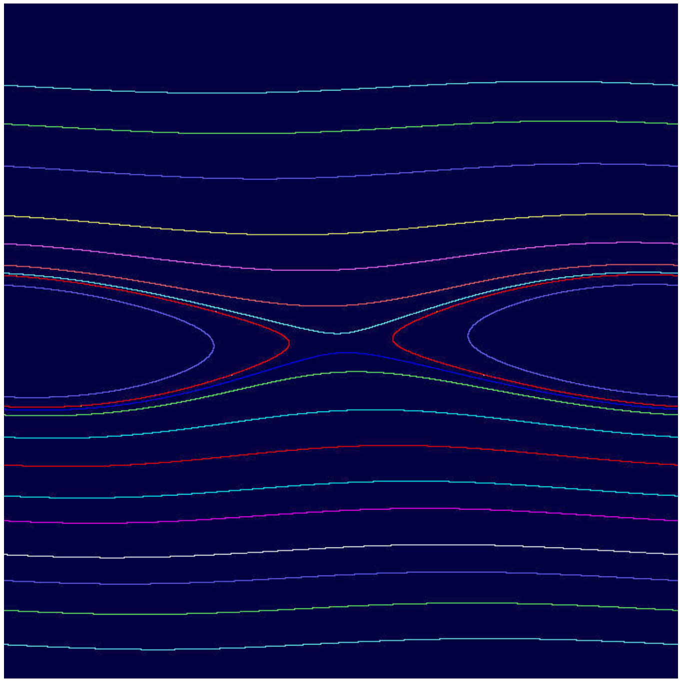
The case of one degree of freedom

If $d = 1$, the motion is bounded. This was seen in In the late 50's in the work of Kolmogorov Arnol'd and Moser for small forcing they proved the first result .

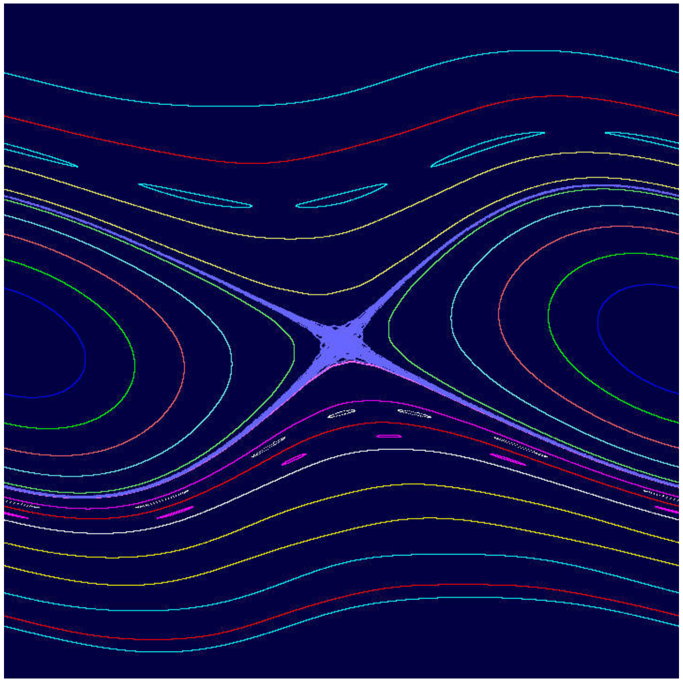
They showed that sufficiently smooth periodic perturbations of an *integrable system*, leave invariant certain sets that are a barrier to gains of energy if $d = 1$.

Diffusion is not possible

The case of one degree of freedom



Time 1 map for perturbed pendulum forcing 0.1



Time 1 map for perturbed pendulum forcing 0.5

The case of two or more degrees of freedom

The previous argument does not work for periodic external perturbations of systems of two or more degrees of freedom.

The KAM tori (d+1-dimensional) do not separate the phase space (2d+1-dimensional).

$2d + 1 - (d + 1) > 1$

A more detailed analysis of Nekhoroshev does not exclude eventual change, but shows that, when the kinetic energy is *convex* in the momenta, the change is slow.

The case of two or more degrees of freedom

The main conjecture:

“Typical systems in action-angle variables have orbits whose actions change widely even if the systems are close to integrable”

Evidences:

- Mathematical:

  An example due to Arnol’d 64 (to be discussed later)

- Numerical studies (Chirikov, Tennyson, Lieberman 75’ on)
- Physical intuition (Fermi, ’34 on)

Main Goals

Can we distinguish when perturbations accumulate and when they do not?

- Given a concrete system, can we say whether the perturbations accumulate or not?
- Can we design systems for which the perturbations accumulate (e.g satellites that use the gravitational energy to move...)
- Can we design systems for which the perturbations do not accumulate (particle accelerators, plasma devices...)

Poincaré program to analyze dynamical systems

Given a concrete dynamical system:

1. Find landmarks that organize the long-term behaviour (periodic orbits, invariant manifolds, homoclinic orbits, KAM tori...
2. Perform a local analysis around them (normal forms, linearizations,....)
3. Study how all this fits together (topology)

We obtain an skeleton of the dynamics In particular, we may obtain regions of instability

Poincaré program to analyze dynamical systems

In this talk, we will describe several combinations of invariant objects and their connections which

- Lead to large effects.
- Are persistent.
- Happen in near integrable systems.
- There are efficient algorithms to compute them.

Poincaré program to analyze dynamical systems

The method of study that we will propose will require to identify *"roads"* in phase space in which the orbits move easily.

We will identify several combinations of objects which lead to diffusion. i.e. different mechanisms with different geometric intuition and different quantitative properties.

New and old Tools

Main tools we will use are standard tools accumulated over many years:

- Persistence of normally hyperbolic manifolds.
- Averaging methods
- KAM theory
- Obstruction properties of manifolds (The λ Lemma).

And new ones:

- The *“scattering map”*
- Correctly aligned windows (M. Gidea and R. de la LLave)

Warning: The effects considered happen only in ≥ 5 dimensions, so it will require some imagination in the presentation.

Other methods that produce results

Variational methods have been producing important results. The techniques we propose can be intermixed with variational and other theories.

Announces by Mather (with detailed sketches) Xia.

One can find proofs of other mechanisms in papers by:

Bolotin-Treschev (1999), Moeckel (1996, 2002), Easton-Meiss-Roberts (2001), Contreras-Paternain (2002), Treschev(2002, 2004), Cheng-Yan (2004), Marco-Sauzin (2002), Cresson-Guillet (2003), Berti-Bolle (2002), Berti-Biasco-Bolle (2003), Bessi-Chierchia-Valdinoci (2001), Bernard (2004), Kaloshin(2003), Bourgain-Kaloshin (2004), De la LLave-Guidea (2005), Kaloshin-Mather-Valdinocci (2004) Kaloshin-Levi (2008).

Best known example in the mathematical literature: Arnol'd example:

Theorem

Let H be

$$\begin{aligned} H(I,\varphi,t;\varepsilon,\mu) &= \frac{1}{2}(I_1^2+I_2^2)+\varepsilon(\cos\varphi_1-1) \\ &\quad +\mu\varepsilon(\cos\varphi_1-1)\,(\sin\varphi_2+\cos t)\ ,\ 0<\mu\ll\varepsilon\ll 1. \end{aligned}$$

Then, there exist orbits of the Hamilton's equations with

$$|I(T)-I(0)|>1\ .$$

Geometric idea of the Arnol'd example

The phase space is 5 dimensional: $\mathbb{R}^2 \times \mathbb{T}^3$.

$$\varepsilon = 0\text{: } H(I, \varphi, t; 0, 0) = \frac{1}{2}(I_1^2 + I_2^2)$$

Integrable system. The solutions are in 3d-tori:

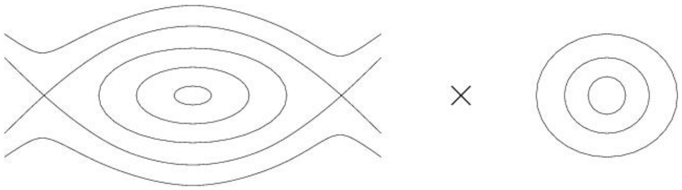
$$I_1 = I_1^0,\ I_2 = I_2^0,\ (\varphi_1, \varphi_2, t) \in \mathbb{T}^3$$

Geometric idea of the Arnol'd example

$$\varepsilon > 0,\ \mu = 0\text{: } H(I, \varphi, t; \varepsilon, 0) = \frac{1}{2}(I_1^2 + I_2^2) + \varepsilon(\cos\varphi_1 - 1).$$

Integrable system (model of a simple resonance)

Some 3 dimensional tori survive (KAM): they correspond to the rotation orbits in the pendulum.



Other tori are destroyed given rise to whiskered two-dimensional tori.

They are hyperbolic tori whose three-dimensional stable and unstable manifolds (whiskers) coincide along a homoclinic manifold.

Diffusion mechanism when $\varepsilon > 0$, $\mu > 0$.

$$\begin{aligned} H(I,\varphi,t;\varepsilon,\mu) &= \frac{1}{2}(I_1^2+I_2^2)+\varepsilon(\cos\varphi_1-1) \\ &\quad +\mu\varepsilon(\cos\varphi_1-1)\left(\sin\varphi_2+\cos t\right),\ 0<\mu\ll\varepsilon\ll 1. \end{aligned}$$

- In particular, all the whiskered tori are preserved we turn on μ.

Diffusion mechanism when $\varepsilon > 0$, $\mu > 0$.

For $\varepsilon > 0$, $\mu > 0$.

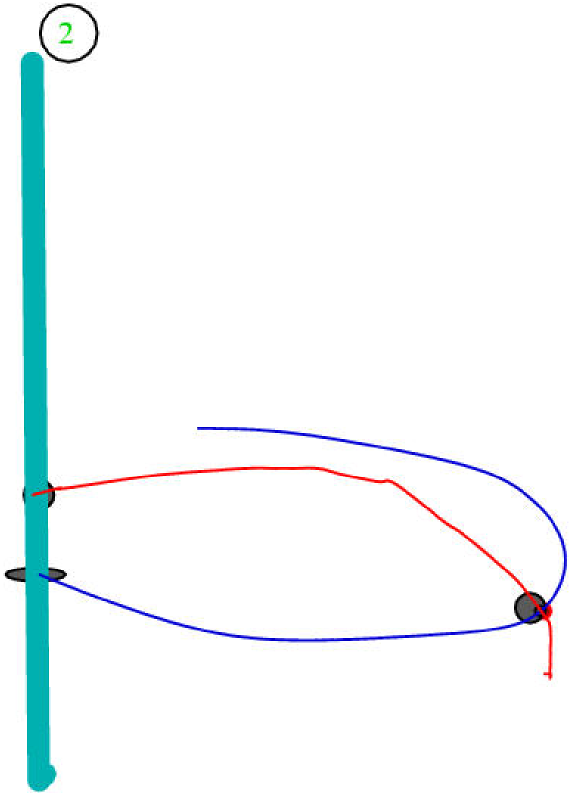
- On the other hand, turning on μ makes the whiskers change. The computation of this change is the Poincaré, Melnikov method.

Note for experts The calculation is somewhat tricky because the main term in μ is $\mu e^{-C\varepsilon^{-1/2}}$.

Nevertheless, if $\mu \ll \varepsilon$ it is possible to prove that the calculation in first order is enough.

The stable and unstable manifolds of every torus intersect transversaly along a homoclinic orbit.

Diffusion mechanism when $\varepsilon > 0$, $\mu > 0$.

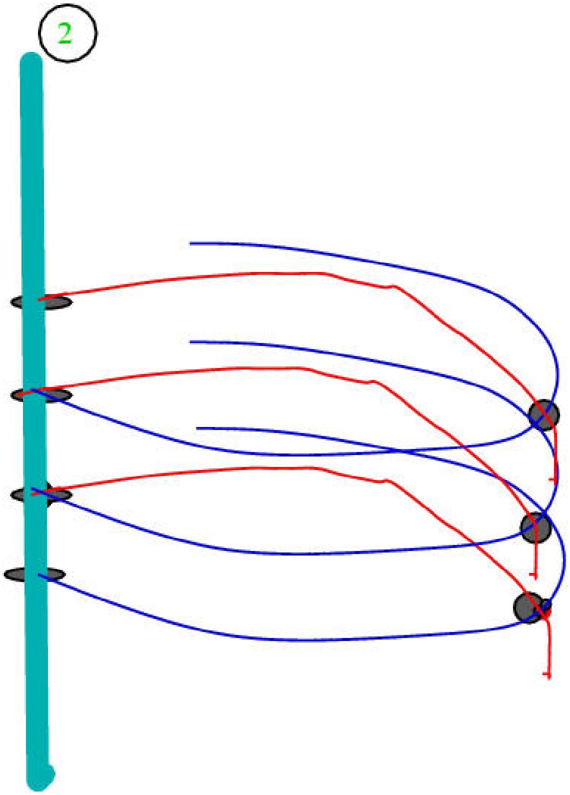


Transversal homoclinic orbits give rise to transversal heteroclinic orbits between tori sufficiently close.

The unstable whisker of a torus intersects transversally the stable whisker of another neighboring torus.

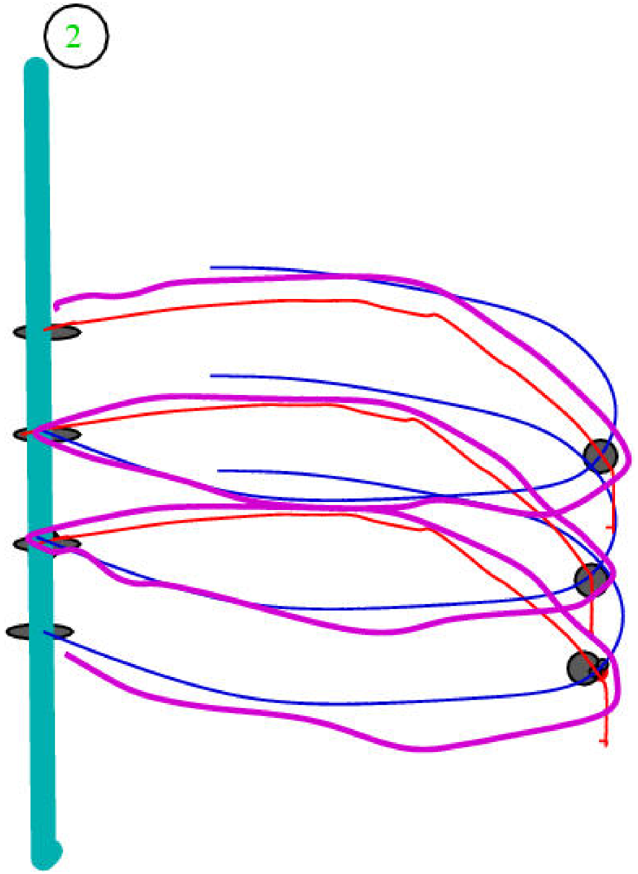
Diffusion mechanism when $\varepsilon > 0$, $\mu > 0$.

We find $\{\mathbb{T}_i\}_{i=1}^N$ such that $W^u_{\mathbb{T}_i} \pitchfork W^s_{\mathbb{T}_{i+1}}$. (*transition chain*.)



Diffusion mechanism when $\varepsilon > 0, \mu > 0$.

There is an orbit that shadows the transition chain. (*obstruction property*)



Implementations of Arnol’d strategy:

Given an integrable system, and an analytic ball around it, there exists one system which experiences diffusion.

R. Douady –P. Le Calvez (neighborhoods of elliptic points)

E. Fontich– P. Martin

Important parts of the strategy

L. Chierchia, G. Gallavotti

(Of course, there are other strategies at play to produce diffusion, notably variational, corrected aligned windows.)

First observation: "a priori unstable systems"

The rigorous verification of Arnol'd mechanism uses the condition $\mu = O(e^{-C\varepsilon^{-1}})$ (still open for $\mu = O(\varepsilon^p)$, M. Guardia)

P. Holmes, J. Marsden (1982): $\varepsilon = 1,\ 0 < \mu \ll 1$

$$H(I, \varphi, t; \mu) = \frac{1}{2}(I_1^2 + I_2^2) + \cos\varphi_1 - 1 + \mu(\cos\varphi_1 - 1)(\sin\varphi_2 + \cos t)$$

Gallavotti:

a priori unstable system

$$H(I, \varphi, t; \varepsilon) = \frac{1}{2}(I_1^2 + I_2^2) + \cos\varphi_1 - 1 + \varepsilon h(I, \varphi, t; \varepsilon)$$

Second observation: the “large gap”problem.

Even in the a priori unstable system case, the Arnol’d example is based on the fact that all the 2-dimensional (hyperbolic) tori are preserved.

- In general, $\mathbb{T}_{I_2}$ can be destroyed when $\varepsilon \neq 0$; KAM Theorem. Large gaps are typical.
- The gaps between the tori that survive are balls of radius $\sqrt{\varepsilon}$ centered in the resonances ($I_2 = m/n$).
- The heteroclínic jumps are of order ε.
- Evidence (numerical, heuristic) suggests that diffusion is stronger precisely on resonances (Chirikov, Tennyson, Simó, Laskar.....).

  Arnold mechanism can not be applied to general perturbations of a priori unstable systems.

A mechanism of gain of instability.

Several models

- Models with two time scales: The Mather model (geodesic flows with periodic or quasi-periodic potentials).

  Small gaps: Arnold mechanism works.

- The restricted elliptic plane three body problem: Model without gaps. The whiskers ”move”a quantity which is exponentially small (as in Arnold model)

- The restricted circular three body problem: Canalias, Delshams, Masdemont, Gidea, Roldan: numerical implementations of the mechanisms and seminumerical proofs.

- The big gaps model.

The big gaps model.

- Two and a half degrees of freedom:

  overcoming the large gaps problem: when there are resonances, they destroy the primary tori, but create secondary tori that can be used to produce transition chains.

- Higher dimensions:

  besides simple resonances, which create secondary tori, there are multiple resonances: the mechanism proposed is to avoid these resonances which are of higher codimension.

(quasi)-periodic perturbations of geodesic flows.

M a n-dimensional manifold

g a $\mathcal{C}^r$ metric on it (r sufficiently large).

We assume:

H1 There exists a closed geodesic "Λ" such that its corresponding periodic orbit $\hat{\Lambda}$ under the geodesic flow is hyperbolic.

H2 There exists another geodesic "γ" such that $\hat{\gamma}$ is a transversal homoclinic orbit to $\hat{\Lambda}$.

That is, $\hat{\gamma}$ is contained in the intersection of the stable and unstable manifolds of $\hat{\Lambda}$, $W^s_{\hat{\Lambda}}$, $W^u_{\hat{\Lambda}}$, in the unit tangent bundle.

Moreover, we assume that the intersection of the stable and unstable manifolds of $\hat{\Lambda}$ is transversal along $\hat{\gamma}$.

(quasi)-periodic perturbations of geodesic flows.

Hipotheses **H1**, **H2** are abundant.

Hedlund, Morse, Mather (generic on the $\mathbb{T}^2$)

Contreras (generic in $\mathcal{C}^2$ topology for any compact manifold)

Katok (true any surfaces of genus bigger or equal than 2, if $r \geq 2 + \delta$, $\delta > 0$.)

(quasi)-periodic perturbations of geodesic flows.

Theorem (Delshams-LLave -S)

Let $\nu \in \mathbb{R}^d$ be Diophantine, $r \in \mathbb{N}$ be sufficiently large (depending on τ, the Diophantine exponent of ν).

Let g be a $\mathcal{C}^r$ metric on a compact manifold M, verifying hypotheses **H1**, **H2**, and $U : M \times \mathbb{T}^d \to \mathbb{R}$ a generic $\mathcal{C}^r$ function.

Consider the time dependent Lagrangian

$$L(q, \dot{q}, \nu t) = \frac{1}{2} g^q(\dot{q}, \dot{q}) - U(q, \nu t), \tag{1}$$

where g^q denotes the metric in $\mathbf{T}_q M$.

Then, the Euler-Lagrange equation of L has a solution $q(t)$ whose energy

$$E(t) = \frac{1}{2} g^q(\dot{q}(t), \dot{q}(t)) + U(q(t), \nu t),$$

tends to infinity as $t \to \infty$.

Restricted elliptic three body problem.

The classical three–body problem studies the dynamics of 3 point masses, denoted $q_1,\ q_2,\ q_3$, in the plane $\mathbb{R}^2$, attracted under Newtonian gravitation.

Restricted three body problem: assume that $m_3 = 0$ and call $\mu = m_2/(m_1 + m_2)$, the mass ratio $0 < \mu < 1$ (Sun–Jupiter–asteroid or comet).

q_1 and q_2 are the primaries. They move in elliptic orbits with excentricity e_0, about their center of mass.

The model

Time-periodic Hamiltonian system (2 and 1/2 degrees of freedom):

$$H_\mu(q,p,t,e_0) = \frac{p^2}{2} - U_\mu(q,\phi,t+t_0,e_0)$$

where

$$U_\mu(q,t+t_0,e_0) = \frac{(1-\mu)}{|q-q_1(t,e_0)|} + \frac{\mu}{|q-q_2(t,e_0)|}$$

Polar coordinates:

$$H_\mu(r,\phi,P_r,G,t,e_0) = \frac{P_r^2}{2} + \frac{G^2}{2r^2} - U_\mu(r,\phi,t+t_0,e_0)$$

When $\mu = 0$, the angular momentum G is constant

Diffusion in the RP3BP

Theorem (Delshams-Kaloshin-S)

Let $0 < e_0 < 0.25$ and $G_0 > 10$.

Then for any small $\mu > 0$ there is $G_1 = G_1(\mu) > G_0$ such that $G_1(\mu) \to \infty$ as $\mu \to 0$ with the property that there is an orbit of the Comet $q(t)$ whose angular momentum $G(t)$ and eccentricity $e(t)$ satisfy:

- Nearly parabolic: $|e(t) - 1| < C\mu$ for any $t \in \mathbb{R}$;
- Presence of diffusion:

in the future

$$\limsup_{t \to +\infty} G(t) > G_1$$

and in the past

$$\liminf_{t \to -\infty} G(t) < G_0$$

.

A priory unstable system: large gap problem

$$H_\varepsilon(p,q,I,\varphi,t) = \frac{1}{2}\,p^2 + V(q) + \frac{1}{2}\,I^2 + \varepsilon\, h(p,I,\varphi,t)$$

Theorem (Delshams-LLave-S)

Assume that

1) h is differentiable enough;
2) $V : \mathbb{T} \to \mathbb{R}$ has a non-degenerate global maximum ($V'(0) = 0$, $V''(0) < 0$).
3) h is a trigonometric polynomial in φ, t;

   (eliminated by Delshams-Huguet and Gidea-Llave)

4) h satisfies appropriate non-degeneracy conditions that can be checked explicitly.

Then, given any interval $[I_-, I_+]$ (which can include resonances) there exists an orbit of the system such that

$$I(0) < I_-, \quad I(T) > I_+ \ .$$

example $H = \pm(\frac{p^2}{2} + \cos q - 1) + \frac{I^2}{2} + \varepsilon \cos q g(\varphi, t)$,

$g(\varphi, t) = a_0 \cos\varphi + a_1 \cos(\varphi - t)$, $a_0 a_1 (a_0^2 - a_1^2) \neq 0$

Verifies 1)...5) in $[-1/2, 3/2]$.

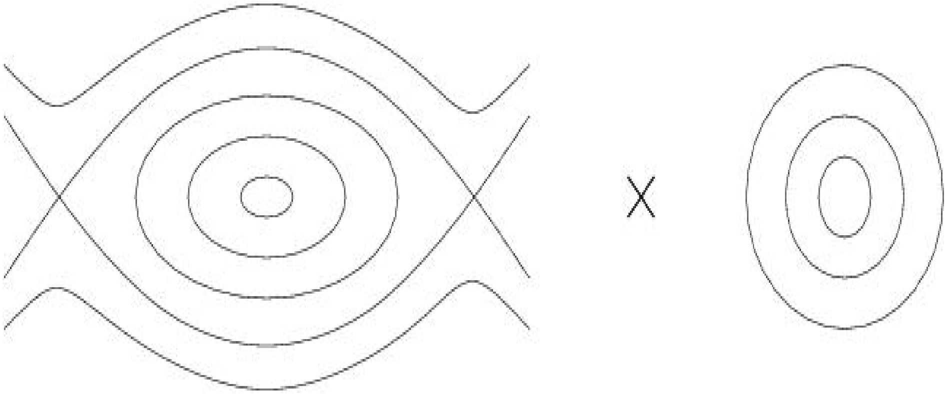
The method: crossing the gaps through secondary tori

Heuristic idea: When there are resonances, they destroy the primary tori, but create other objects that can be used to produce transition chains.

- secondary tori: tori of different topology: retractable to a periodic orbit.

To find these objects we need to change the strategy of the proof.

The unperturbed problem ($\varepsilon = 0$): the normally hyperbolic invariant manifold

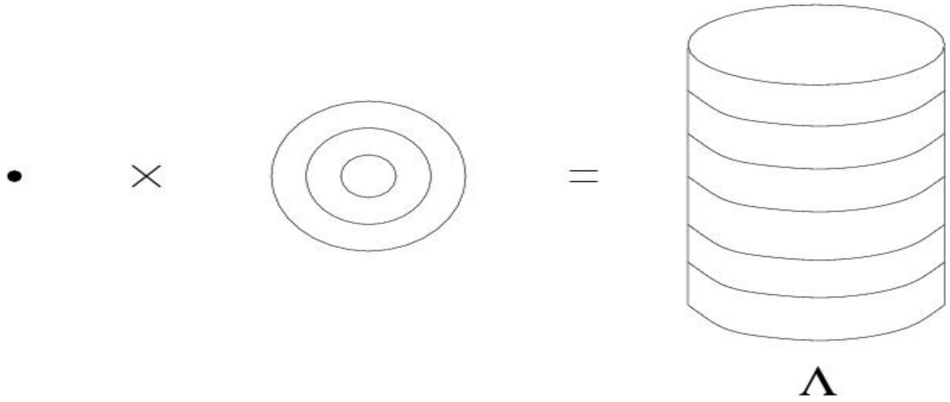


- $H_0(p, q, I, \varphi) = \frac{p^2}{2} + V(q) + \frac{I^2}{2}$ is integrable.
- For any $I \in \mathbb{R}$ the two dimensional torus

$$\mathbb{T}_I^0 = \{(p, q, I, \varphi, t) : \; p = q = 0, \; (\varphi, t) \in \mathbb{T}^2\}$$

is invariant and its stable and unstable 3-dimensional manifolds coincide

The unperturbed problem ($\varepsilon = 0$): the normally hyperbolic invariant manifold



- $\tilde{\Lambda} \sim \mathbb{R} \times \mathbb{T}^2$ is a 3-dimensional normally hyperbolic invariant manifold filled by invariant tori.
- (I, φ, t) are good global coordinates in $\tilde{\Lambda}$

The unperturbed problem ($\varepsilon = 0$): the scattering map

- The unstable and stable 4-dimensional manifolds of $\tilde{\Lambda}$ coincide along a homoclinic manifold $\tilde{\gamma}$
- Sccatering map associated to the homolinic manifold $\tilde{\gamma}$.

  $S_0 : \tilde{\Lambda} \to \tilde{\Lambda}$

  $x_+ = S_0(x_-)$

  when

$$W^s(x_+) \cap W^u(x_-) \neq \emptyset$$

The unperturbed problem ($\varepsilon = 0$): the scattering map

When $\varepsilon = 0$ we have $S_0 = Id$, so that

$$S_0(\mathbb{T}_I^0) = \mathbb{T}_I^0$$

The unperturbed tori only have homoclinic connexions.

Main goal:

We want to see that, when $\varepsilon \neq 0$ we can define a scattering map such that, the image of one torus intersects other tori.

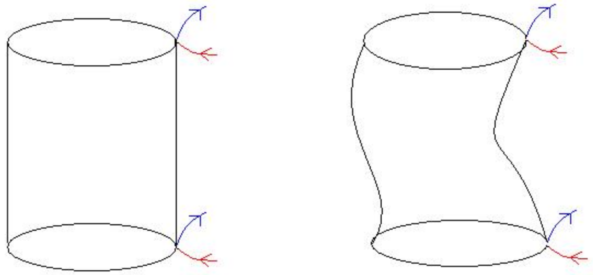
Sketch of the proof:

1) Persistence of $\tilde{\Lambda}$.
2) Study of the dynamics on $\tilde{\Lambda}_\varepsilon$.
3) Study of stable and unstable manifolds for $\tilde{\Lambda}_\varepsilon$.
4) The scattering map.
5) Transition chains.
6) Shadowing lemma.

The perturbed problem ($\varepsilon \neq 0$): persistence of $\tilde{\Lambda}$

- Observe that it is a normally hyperbolic manifold and apply Fenichel, Hirsch-PughShub theory.
- We obtain an exact symplectic manifold $\tilde{\Lambda}_\varepsilon$ wich is ε close to $\tilde{\Lambda}$.
- We can obtain a good system of coordinates in $\tilde{\Lambda}_\varepsilon$: (I, φ, t) such that the flow restricted to it is exact symplectic.
- The stable and unstable manifolds $W^{s,\mathrm{loc}}(\tilde{\Lambda}_\varepsilon)$ of $\tilde{\Lambda}_\varepsilon$ are close to the unperturbed ones.

The perturbed problem ($\varepsilon \neq 0$): persistence of $\tilde{\Lambda}$



The perturbed problem ($\varepsilon \neq 0$): dynamics in $\tilde{\Lambda}_\varepsilon$

The Hamiltonian generating the flow in $\tilde{\Lambda}_\varepsilon$ has the form:

$$k(J, \varphi, t; \varepsilon) = \frac{J^2}{2} + \varepsilon k_1(J, \varphi, t) + \varepsilon^2 k_2(J, \varphi, t) + \cdots$$

$k_i(J, \varphi, t)$ are computable explicitly from h.

The perturbed problem ($\varepsilon \neq 0$): dynamics in $\tilde{\Lambda}_\varepsilon$: averaging

Given a Hamiltonian

$K(J, \varphi, t; \varepsilon) = \frac{J^2}{2} + \varepsilon H(J, \varphi, t)$,

where $H(J, \varphi, t)$ is periodic:

$H(I, \varphi, t) = \sum_{(k,n) \in \mathcal{N}} H_{k,n}(I) e^{i(k\varphi + nt)}$

Look for a change of variables that reduce the system to motion of the actions to constant up to any order in ε

$\bar{K}(J, \varphi, t; \varepsilon) = \frac{J^2}{2} + \varepsilon H_0(J) + O(\varepsilon^n)$.

The perturbed problem ($\varepsilon \neq 0$): dynamics in $\tilde{\Lambda}_\varepsilon$: averaging

Resonances of first order:

$kI + n = 0$; that is $I = -n/k$, for $(n, k) \in \mathcal{N}$

Far from resonances we obtain

$$\frac{I^2}{2} + \varepsilon h_0(I; \varepsilon) + O(\varepsilon^n)$$

Close to resonances $I = -n_0/k_0$ we obtain:

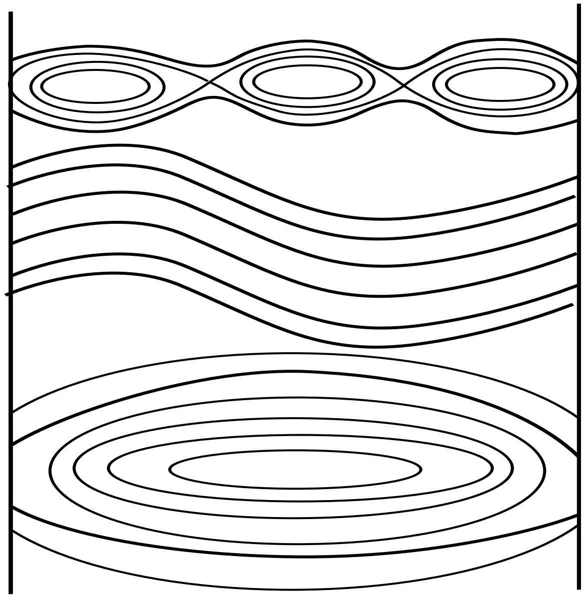
$$\frac{I^2}{2} + \varepsilon h_0(I; \varepsilon) + \varepsilon V(k_0\theta + n_0 t) + O(\varepsilon^n)$$

Motion is pendulum like

(See the book of Arnold, Kozlov, Neishtadt *Dynamical Systems III*)

The perturbed problem ($\varepsilon \neq 0$): dynamics in $\tilde{\Lambda}_\varepsilon$: averaging

The motion of the averaged system is:



The perturbed problem ($\varepsilon \neq 0$): dynamics in $\tilde{\Lambda}_\varepsilon$: KAM

The averaged system gives us a very approximated idea of the dynamics in $\tilde{\Lambda}_\varepsilon$ and some approximated equations for the tori.

KAM Theory

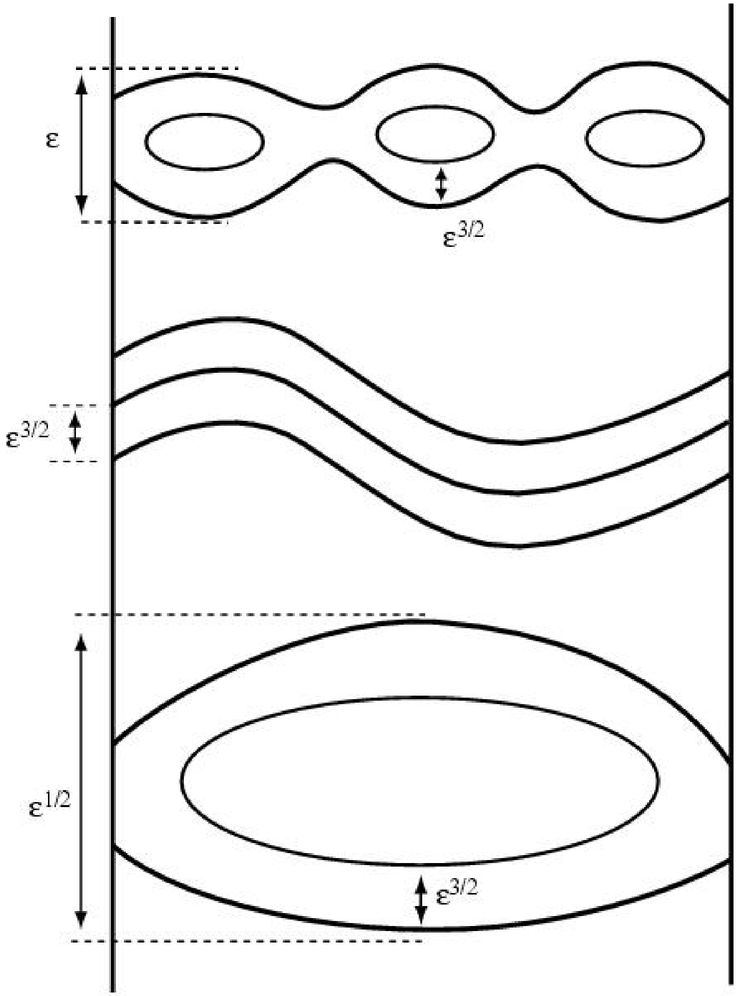
If a rotation vector is hard to approximate by rational numbers, then tori with this rotation persist under smooth enough perturbations.

Near approximately invariant tori with Diophantine rotations, there are exactly invariant tori.

See `www.ma.utexas.edu/mp_arc` Paper 01-29 for a tutorial.

The perturbed problem ($\varepsilon \neq 0$): dynamics in $\tilde{\Lambda}_\varepsilon$: KAM

Applying constructive versions of KAM we obtain



The perturbed problem ($\varepsilon \neq 0$): dynamics in $\tilde{\Lambda}_\varepsilon$: some comments

- The averaging procedure and the proximity of primary and secondary tori has been studied extensively by many people, including A. Neishtadt (Lochak-Meunier), Kozlov, etc.

  For very differentiable systems, the distance between the tori and secondary tori can be made to be a larger power of ϵ. For analytic systems, it can be made exponentially small. (Normal hyperbolic manifolds are not analytic, so this last part is not of use to us.)

- The fact that we can fill the regions without resonances by close KAM tori is related to "gap bridging mechanism" of Chierchia-Gallavotti.)

The perturbed problem ($\varepsilon \neq 0$): dynamics in $\tilde{\Lambda}_\varepsilon$: some comments

- Justifying the pictures involves some assumptions that say that certain coefficients are not zero.
- If h is a trigonometric polynomial, only a finite number of resonances. (This has been eliminated, Delshams-Huguet).
- One needs to obtain the equations of the tori in the original variables
- Be sure that the resonant zones and the non-resonant ones overlap: last secondary torus sufficiently close to the first primary torus.

The perturbed problem ($\varepsilon \neq 0$): the stable and unstable manifolds of $\tilde{\Lambda}_\varepsilon$

The stable and unstable manifolds of Λ_ε split, they intersect transversally.

- Computing the first order variational equations we obtain a Melnikov potential $\mathcal{L}(I, \varphi, t)$.
- The nondegeneracy assumption in hypothesis H4 gives the existence of non-degenerate critical points of the function $\mathcal{L}(I, \varphi - I\tau, t - \tau)$.
- Associated to these critical points τ^* there exist transversal homoclinic orbits to Λ_ε.

The perturbed problem ($\varepsilon \neq 0$): the scattering map in $\tilde{\Lambda}_\varepsilon$

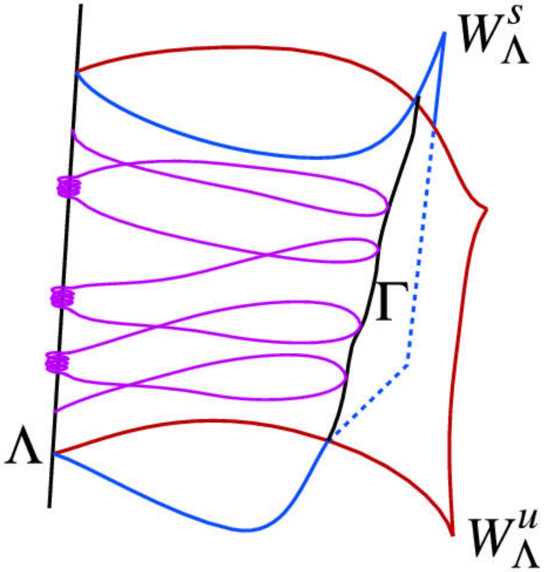
The existence of transversal homoclinic orbits allows us to define the perturbed scattering map. Choose a homoclinic family of orbits

$$\Gamma \subset W^s_{\Lambda_\varepsilon} \cap W^u_{\Lambda_\varepsilon}$$

We associate a scattering map $x^+ = S_\varepsilon(x^-)$ if

$\exists z \in \Gamma$ s.t. $z \in W^s_{x_+} \cap W^u_{x_-}$.

The perturbed problem ($\varepsilon \neq 0$): the scattering map in $\tilde{\Lambda}_\varepsilon$



We can obtain perturbative formulas for it

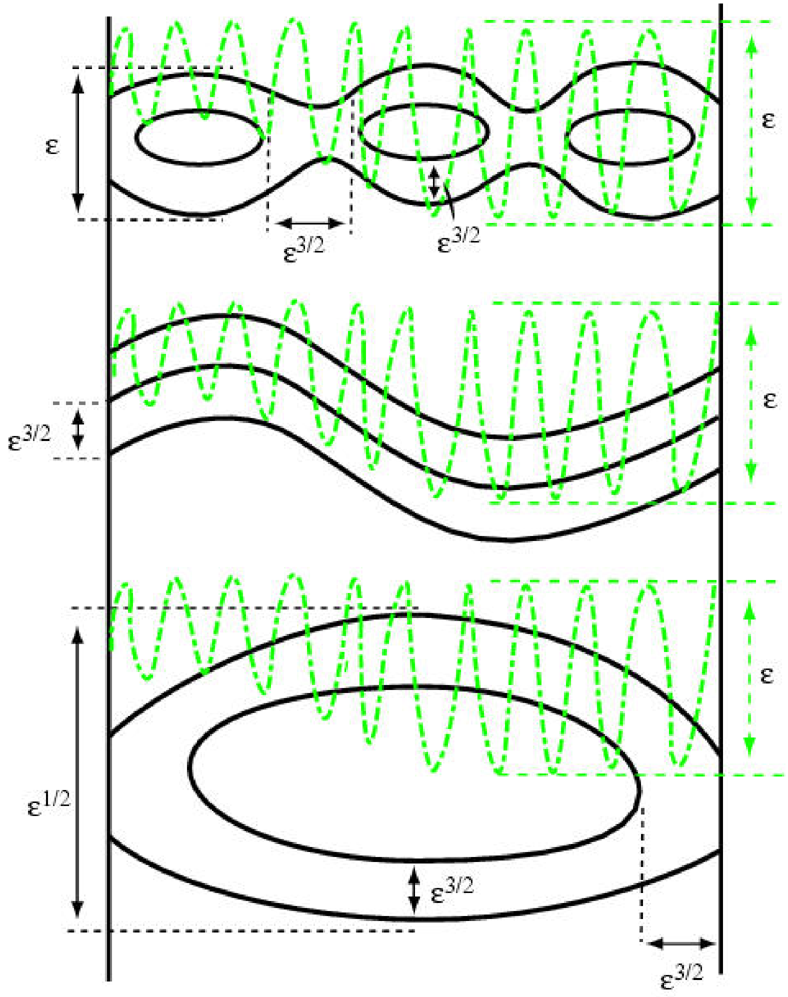
$$I_+ = I_- - \varepsilon \frac{\partial}{\partial \varphi} \mathcal{L}(I, \varphi - I\tau, t - \tau) + O(\varepsilon^{1+c_2}),\ c_2 > 0,$$

The perturbed problem ($\varepsilon \neq 0$): the scattering map in $\tilde{\Lambda}_\varepsilon$

- The scattering map is defined in open, non-empty sets. In some examples, the whole manifold.
- It is exact symplectic.
- It is a substitute for the usual Melnikov functions for invariant tori. It can be applied to any point in $\tilde{\Lambda}_\varepsilon$, not only in invariant objects.
- With the scattering map, we are able to find heteroclinic connections between objects of different topologycal type.

The perturbed problem ($\varepsilon \neq 0$): transition chains

When we apply the scattering map to the objects we had identified in $\tilde{\Lambda}_\varepsilon$ we obtain



The perturbed problem ($\varepsilon \neq 0$): transition chains

A moment's reflection unraveling the definition of the scattering map shows that

Proposition If $S(\mathcal{H}) \pitchfork_{\Lambda} \mathcal{G}$, then, $W^u_{\mathcal{H}} \pitchfork W^s_{\mathcal{G}}$.

Therefore, we can obtain transition chains mixing primary tori, and secondary tori.

The perturbed problem ($\varepsilon \neq 0$): Shadowing orbits

Fontich-Martin (2003):

Lema Given a transition chain $\{\mathbb{T}_i\}_{i=0}^{N}$. Given a sequence $(\delta_i)_{i=0,\ldots,N}$, one can find a point $x \in (\mathbb{R} \times \mathbb{T})^2 \times \mathbb{T}$ and a sequence of numbers $0 = t_0 < \cdots < t_N$ such that

$$\phi_{t_i}(x) \in \mathbf{B}(\mathbb{T}_i, \delta_i)$$

It can be used because it does not requires that the connected objects have the same topological type or the same dimension.

Higher dimension, the model.

$$\begin{aligned} H(I,\varphi,p,q,t,\varepsilon) &= h_0(I) + \sum_{j=1}^{n} \pm(\frac{1}{2}p_j^2 + V_j(q_j)) \\ &+ \varepsilon h_1(I,\varphi,p,q,t,\varepsilon) \end{aligned}$$

where $I \in \mathbb{I} \subset \mathbb{R}^d, \varphi \in \mathbb{T}^d$, $p, q \in \mathbb{R}^n$, $t \in \mathbb{T}^1$

Under similar hypothesis plus:

The mapping $I \to \frac{\partial}{\partial I} h_0(I) \equiv \omega(I)$ is a diffeomorphism of $\mathbb{I}$ to its image.

Higher dimension: the normally hyperbolic invariant manifold.

The normally hyperbolic invariant manifold $\tilde{\Lambda}_\varepsilon$ $2d+1$ dimensional.

The flow in $\tilde{\Lambda}_\varepsilon$ is Hamiltonian:

$$K(I,\varphi,t,\varepsilon) = h(I) + \sum_{i=1}^{N} \varepsilon^i R_i(I,\varphi,t) + O(\varepsilon^{N+1})$$

Higher dimension: resonances.

- $d = 1$ the set of resonant actions is a finite set of rational numbers $I = k/l$

- In general, $I \in \mathbb{I} \subset \mathbb{R}^d$ is resonant if

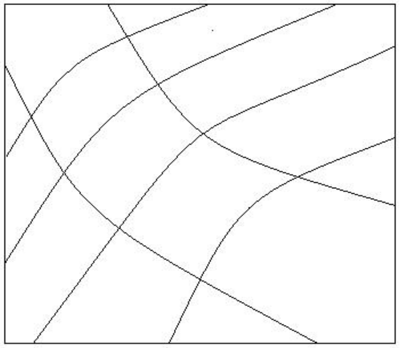
$$\omega(I) \cdot k + l = 0$$

for some $(k, l) \in \mathbb{Z}^d \times \mathbb{Z}$, where $\omega(I) = \nabla h_0(I)$.

This is a hypersurface in $\mathbb{I}$.

Higher dimension: multiple resonances

The case $d \geq 2$ presents a difficulty that was not present in the case $d = 1$, namely, that, in the action space, there are points where the resonances have higher multiplicity, that is, points that lay in the intersection of two hypersurfaces.



Higher dimension: averaging

- Far from resonances we obtain a Hamiltonian

$$\bar{h}(I) + O(\varepsilon^{N+1})$$

- Close to a simple resonance (k_0, l_0), we can adapt the averaging method to obtain a hamiltonian or the form

$$\bar{h}(I) + \varepsilon U(I_1, \cdots, I_{d-1}, \varphi) + O(\varepsilon^{N+1})$$

  The dynamics is like a pendulum and $d - 1$ rotators uncoupled.

  We obtain again primary and secondary tori

- Close to a multiple resonance, the averaged system is not integrable!

Higher dimension: avoiding multiple resonances

We will adapt the previous methods to show instability under rather explicit conditions.

The basic observation is that multiple resonances happen in subsets of codimension greater than 1 in $\mathbb{I}$. Hence, we will use the methods of the previous case to contour these sets so that the diffusing orbits can proceed.

Higher dimension.

Theorem(Delshams-LLave-S)

For any C^1 curve γ parameterized by a function $\hat{\gamma} : [0, \infty] \longrightarrow \mathbb{I}^*$ such that:

- γ does not intersect the multiples resonances
- The intersection of γ with the resonant lines is transversal.

There exists $\varepsilon_0 > 0$, $C > 0$ such that for every $0 < |\varepsilon| < \varepsilon_0$, there exists an orbit $x(t)$ and a monotone increasing function Γ such that:

$$|I(x(t)) - \hat{\gamma}(\Gamma(t)| \leq C\varepsilon^{1/2}$$

(quasi)-periodic perturbations of geodesic flows

Hamiltonian formulation

$$H(q,p,\nu t) = \frac{1}{2} g_q(p,p) + U(q,\nu t),$$

where g_q is the metric in $\mathbf{T}^*M$.

Orbits of the geodesic flow ($U = 0$) rescale with energy as

$$(\lambda_E^p(t), \lambda_E^q(t)) = \left(\sqrt{2E}\lambda_{1/2}^p\left(\sqrt{2E}t\right), \lambda_{1/2}^q\left(\sqrt{2E}t\right)\right).$$

(quasi)-periodic perturbations of geodesic flows

When $U \neq 0$:

for high energy, the external potential is a small (and slow) perturbation of the extended flow

We pick a (large) number $E^* > 0$ and introduce

$$\varepsilon = 1/\sqrt{E^*}.$$

after some scaling of variables and time the new Hamiltonian

$$H_\varepsilon(q, p, \varepsilon\nu t) = \frac{1}{2} g_q(p, p) + \varepsilon^2 U(q, \varepsilon\nu t),$$

Due to the fact that the potential depends slowly on time:

the gaps between the KAM tori are very small if the system has enough differentiability!

Restricted three body problem

A priori unstable structure

Let $x^2 := 1/r$, $y = P_r$, $G = P_\phi$), new equations:

$$\begin{aligned}
\dot{x} &= -\frac{x^3}{2}\frac{\partial H_G}{\partial y} \\
\dot{y} &= \frac{x^3}{2}\frac{\partial H_G}{\partial x} \\
\dot{\phi} &= \frac{\partial H_G}{\partial G} \\
\dot{G} &= -\frac{\partial H_G}{\partial \phi}
\end{aligned}$$

$$H_G(x, y, \mu) = \frac{y^2}{2} + \frac{G^2 x^4}{2} - x^2 + \mu \tilde{U}(x, \phi, t),$$

One limit: the two body problem: $\mu = 0$

A priori unstable structure: An invariant "normally hyperbolic" cylinder.

- The 3 dimensional manifold:

$$\mathcal{A}_\infty = \{x = y = 0,\ (\phi, G, t) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\}$$

  is invariant and foliated by periodic orbits.

- The inner dynamics on $\mathcal{A}_\infty$ is trivial:

$$(\phi_0, G, t_0) \to (\phi_0, G, t_0 + t)$$

A priori unstable structure: An invariant “normally hyperbolic” cylinder for $\mu = 0$.

- $\mathcal{A}_\infty$ it is not normally hyperbolic with respect to the “real” time t.

  We can introduce a time-reparametrization $t = t_G(\tau) = \frac{G^3}{2}(\sinh\tau + \frac{\sinh^3\tau}{3})$ that makes $\mathcal{A}_\infty$ normally hyperbolic.

- $W_0^s(\mathcal{A}_\infty) = W_0^u(\mathcal{A}_\infty) = \{H_G(x, y) = 0,\ (\phi_0, G, t_0) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\} = \tilde{\gamma}$

The scattering map in $\mathcal{A}_\infty$.

We can define the scattering map in $\mathcal{A}_\infty$ associated to the homolinic manifold $\tilde{\gamma}$

$S_0 : \mathcal{A}_\infty \to \mathcal{A}_\infty$

When $\mu = 0$ we have that $S_0 = Id$,

The unperturbed periodic orbits only have homoclinic connexions.

Main goal:

We want to see that, when $\mu \neq 0$ we can define a scattering map such that, the image of one periodic orbit intersects other periodic orbits.

Advantatges: NO gaps

Difficulties: Melnikov function is exponentially small for G big This needs μ exponentially small respect to $1/G$. (usually called a priory stable).

Challenges:

Crossing through multiple resonances

Good models for multiple resonances.

Look for other mechanisms

Deal with exponentially small phenomena.