Stochasticity in Deterministic Systems

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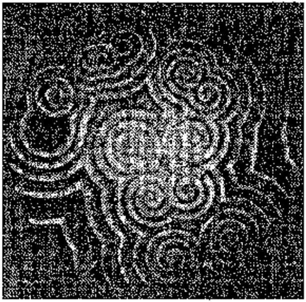
Dynamical Systems and Probability Theory

Dynamical Systems	Probability Theory
Time series $v, v \circ f, v \circ f^2, \ldots$	Sequence of random variables $X_1, X_2, X_3, \ldots$
ID not I	IID (independent & identically distributed)
	Weakly dependent
Ergodic theorem	SLLN (Strong law of large numbers)
	CLT (Central limit theorem)
$\sum_{j=0}^{n-1} v \circ f^j \sim W(n)$	$X_1 + \cdots + X_n \sim W(n)$

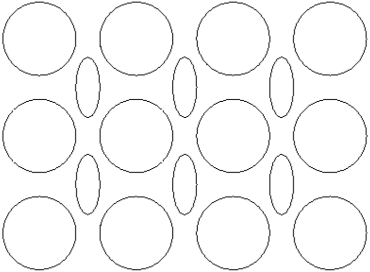
Outline

Deterministic Brownian Motion – Two Examples

Hypermeander of spiral waves in excitable media



Finite horizon planar periodic Lorentz gas



Deterministic Systems that Behave Stochastically

- Hypermeandering spiral waves
- Lorentz gases
- Lorenz attractor
- Hénon attractor
- Logistic family
- Smale horseshoe, solenoid
- Pomeau-Manneville intermittency maps

Classes of Dynamical Systems

- Axiom A: Uniformly expanding maps
  uniformly hyperbolic systems
- Nonuniformly expanding/hyperbolic systems
  in sense of Young
- Singularly hyperbolic flows
- Partially hyperbolic systems

Excitable Media – PDE models

Oregonator

$$\frac{du}{dt} = \frac{1}{\epsilon}\left[u - u^2 - fv(u-q)/(u+q)\right] + D_1 \Delta u$$
$$\frac{dv}{dt} = u - v + D_2 \Delta v$$

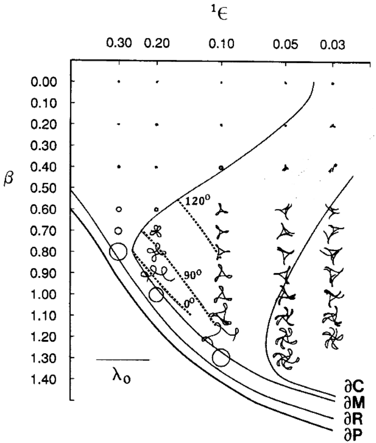
FitzHugh-Nagumo (FHN)

$$\frac{du}{dt} = \frac{1}{\epsilon}\left[u - \frac{1}{3}u^3 - v\right] + D\Delta u$$
$$\frac{dv}{dt} = \epsilon[u + \beta - \gamma v] + \delta D \Delta v$$

Barkley

$$\frac{du}{dt} = \frac{1}{\epsilon}u(1-u)\left(u - \frac{1}{a}(v+b)\right) + \Delta u$$
$$\frac{dv}{dt} = u - v + \delta \Delta v$$

Transitions in the FHN equation



Winfree (1991)
Zykov (1986)

- ∂M: transition from rigidly rotating spirals to meandering spirals
- ∂C: transition from meandering spirals to "complex" states (hypermeander)

Meander Explained

- Meander transition is a Hopf bifurcation

  Barkley, Kness & Tuckerman (1990), Karma (1990)

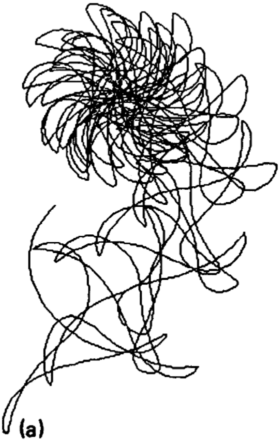
  Experimental verification: Skinner & Swinney (1991)

  Animation: Barkley EZspiral
- Barkley (*PRL 1994*) explained resonance phenomena
  Euclidean symmetry plays fundamental role

  Experimental verification: Li, Ouyang, Petrov & Swinney (*PRL 1996*)

  Rigorous treatment: Wulff 1996, Sandstede, Scheel & Wulff (1997)

Hypermeander in the FHN equation

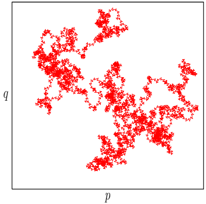


Winfree (1991)
Rössler & Kahlert (1979)

Hypermeander Explained?

- Biktashev & Holden (1998). Deterministic Brownian motion in the hypermeander of spiral waves
  Chaotic dynamics + translation symmetry $\Longrightarrow$ Brownian motion

- Rigorous treatment:
  Ashwin, Field, M,
  Nicol, Török, ...

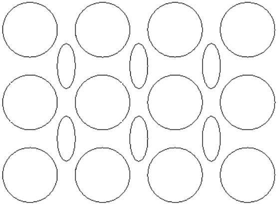


- Chaotic dynamics + rotation symmetry + translation symmetry $\Longrightarrow$ not best example to start with!

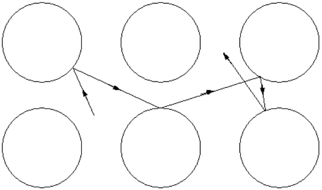
Finite Horizon Planar Periodic Lorentz Gas

- Lorentz 1905. Deterministic model for Brownian motion
- Chernov & Dolgopyat 2006. Open problem at Madrid ICM
- M & Nicol 2009. Solution in *Annals of Probability*
- Chaotic dynamics + translation symmetry $\Longrightarrow$ Brownian motion

Finite and Infinite Horizon



Finite horizon



Infinite horizon

Animation: Sinai billiard Stephan Matthiesen (2006)

Lorentz Gas

- Periodic scatterers $\Omega \subset \mathbb{R}^2$
- 3-dimensional phase space $M = (\mathbb{R}^2 - \Omega) \times S^1$
  Two-dimensional position $p \in \mathbb{R}^2 - \Omega$, unit velocity $v \in S^1$
- Volume preserving flow $t \mapsto (p(t), v(t))$
- Ergodicity studied by Sinai (1972)

Growth of Position Variable

- Sublinear growth $\frac{1}{t}p(t) \to 0$ as $t \to \infty$ for almost every initial condition $p(0), v(0)$
- This is the Strong Law of Large Numbers / Ergodic Theorem
- View velocity $v : M \to \mathbb{R}^2$ as an observable
- $p(t) = \int_0^t v(s)ds$
- Time average $\frac{1}{t}\int_0^t v(s)ds = \frac{1}{t}p(t)$
- Space average $\int_M v = 0$

Theorem (M & Nicol, 2009)

$p(t) = W(t) + O(t^{\frac{7}{15}+\epsilon})$ *almost surely*

$W(t)$ two-dimensional Brownian motion with covariance Σ

$\epsilon > 0$ arbitrarily small

Probability Theory

- $X_1, X_2, \ldots$ sequence of i.i.d. random variables with $EX_1 = 0$, $EX_1^2 = \sigma^2$
- SLLN $\frac{1}{n}(X_1 + \cdots + X_n) \to 0$ almost surely
- CLT $\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) \to_d N(0, \sigma^2)$
  $P(\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) < c) \to \int_{-\infty}^{c} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/2\sigma^2}\, dy$

Deterministic Dynamical System

- $f : M \to M$ map with ergodic probability measure μ
  $v : M \to \mathbb{R}$ integrable, $\int_M v \, d\mu = 0$
  $v, v \circ f, v \circ f^2, \ldots$ are identically distributed but dependent
- SLLN=ergodic theorem $\frac{1}{n}\sum_{j=1}^{n} v \circ f^j \to 0$ almost surely
- $\int_M v^2 \, d\mu < \infty$ does not imply CLT without extra assumptions

Axiom A Systems

- Uniformly expanding maps

Example (Doubling map)

$$f : [0,1] \to [0,1], \quad f(x) = 2x \bmod 1$$

- Uniformly hyperbolic (Axiom A) maps

Example (cat map)

$$f : [0,1]^2 \to [0,1]^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \bmod 1$$

Eg. Smale horseshoe

- Uniformly hyperbolic (Axiom A) flows   Eg. Solenoid

Central Limit Theorem for Axiom A Maps

$f : M \to M$ uniformly expanding/hyperbolic map
$v : M \to \mathbb{R}$ Hölder continuous, $\int_M v \, d\mu = 0$

Theorem (Sinai, Ruelle, Bowen, Ratner)

$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \int_M (\sum_{j=1}^n v \circ f^j)^2 \, d\mu$ *exists*

and $\frac{1}{\sqrt{n}} \sum_{j=1}^n v \circ f^j \to_d N(0, \sigma^2)$

Central Limit Theorem for Axiom A Flows

$f_t : M \to M$ Axiom A flow
$v : M \to \mathbb{R}$ Hölder continuous, $\int_M v \, d\mu = 0$

Theorem (Ratner 1973)

$\frac{1}{\sqrt{t}} \int_0^t v \circ f_s \, ds \to_d N(0, \sigma^2)$ *for some* σ^2

Convergence to Brownian motion

Almost sure invariance principle (ASIP)

- Probability theory, Strassen (1964)
  $X + \cdots + X_n = W(n) + O(n^{\frac{1}{2}-\lambda})$ almost surely
  $W(t)$ Brownian motion with variance σ $\quad \lambda > 0$
  ASIP $\implies \cdots \implies$ CLT and LIL
- Axiom A maps/flows, Hofbauer & Keller (1982), Denker & Philipp (1984)
  $\sum_{j=1}^{n} v \circ f^j = W(n) + O(n^{\frac{1}{2}-\lambda})$ almost surely
  $\int_0^t v \circ f_s \, ds = W(t) + O(t^{\frac{1}{2}-\lambda})$ almost surely

Higher-Dimensional Almost Sure Invariance Principle

- Probability theory, Berkes & Philipp (1979)
  $X + \cdots + X_n = W(n) + O(n^{\frac{1}{2}-\lambda})$ almost surely
  $W(t)$ d-dimensional Brownian motion with covariance Σ
- Axiom A maps/flows, M & Nicol (2009)
  $v : M \to \mathbb{R}^d$ Hölder, $\int_M v = 0$
  $\sum_{j=1}^{n} v \circ f^j = W(n) + O(n^{\frac{2d+3}{4d+7}+\epsilon})$ almost surely
  $\int_0^t v \circ f_s \, ds = W(t) + O(t^{\frac{2d+3}{4d+7}+\epsilon})$ almost surely

For $d = 1$ error term is $O(n^{\frac{1}{4}+\epsilon})$. Field, M & Török (2003)

Lorentz Gas Revisited

Two-dimensional position $p(t)$

Theorem (Bunimovich, Sinai & Chernov 1980s)

1. $\frac{1}{\sqrt{t}}p(t) \to_d N(0, \Sigma)$
2. $\frac{1}{\sqrt{n}}p(nt) \to W(t)$ *weakly in* $C([0,1], \mathbb{R}^2)$

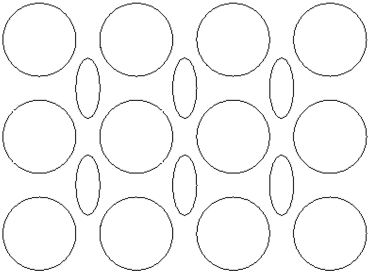
- Σ 2×2 covariance matrix
- $N(0, \Sigma)$ two-dimensional normal distribution
- $W(t)$ Brownian motion with covariance Σ

Part (2): Functional CLT or Weak invariance principle

- ASIP $\implies$ FCLT/WIP $\implies$ CLT
- ASIP $\implies$ FLIL $\implies$ LIL and much more

Decay of Correlations in Billiard Map

Recall $M = (\mathbb{R}^2 - \Omega) \times S^1$
2-dimensional cross-section $X = \partial\Omega \times [-\pi/2, \pi/2]$
Poincaré map $f : X \to X$ called billiard map or collision map



Decay of Correlations in Lorentz gas

Recall $M = (\mathbb{R}^2 - \Omega) \times S^1$
2-dimensional cross-section $X = \partial\Omega \times [-\pi/2, \pi/2]$
Poincaré map $f : X \to X$ called billiard map or collision map

Theorem (Bunimovich, Sinai & Chernov 1980s)

$v, w : X \to \mathbb{R}$ *Hölder observables*

$$|\int_X v \circ f^n w - \int_X v \int_X w| \leq Ce^{-an^\gamma}$$

for constants $C \geq 1$, $a > 0$, $\gamma \in (0, 1]$

Question: $\gamma < 1$ or $\gamma = 1$?

Very controversial for 15-20 years. Numerics suggested $\gamma < 1$

Nonuniform Hyperbolicity

L-S Young (*Annals of Math 1998*) introduced a class of nonuniformly hyperbolic systems modelled by Young towers.

Includes

- Lorentz gas
- Hénon attractor Benedicks & Young (2000)
- Unimodal/multimodal maps

Theorem (Young 1998)

Nonuniformly hyperbolic maps satisfy:
exponential decay of correlations ($\gamma = 1$) and CLT

Theorem (M & Nicol 2009)

d-dimensional ASIP holds for nonuniformly hyperbolic maps and flows

Singular Hyperbolicity

Example (Lorenz equations)

$$\dot{x} = 10(y - z)$$
$$\dot{y} = 28x - y - xz$$
$$\dot{z} = xy - \tfrac{8}{3}z$$

- Not uniform hyperbolic, equilibrium at $(0, 0, 0)$
- Singular hyperbolic (Morales, Pacifico)
- Tucker (1999, 2002) The Lorenz attractor exists
- Logarithmic singularity
- ASIP still holds, Holland & M (2007)

Partial Hyperbolicity

Example (Time one map)

$f = f_1$ where f_t is Axiom A or nonuniformly hyperbolic flow

Example (Compact group extension)

$(x, g) \to (fx, gh(x)), \quad (x, g) \in M \times G, \quad h : M \to G$

- ASIP not known for $\mathbb{R}^d$-valued observables, $d \geq 2$
- $d = 1$, ASIP not known unless there is rapid mixing
- Rapid mixing is typical, Dolgopyat 1998
- Rapid mixing is open and dense, Field, M & Török 2007

Theorem (M & Török 02, Field, M & Török 03)

Rapid mixing $\implies$ ASIP for $d = 1$ and WIP for all d in these examples

Mixing for Flows

Definition

A flow $f_t : M \to M$ is mixing if

$$\mu(f_t(E) \cap F) \to \mu(E)\mu(F) \text{ as } t \to \infty$$

for all measurable $E, F \subset M$

The flow is stably mixing if all nearby flows are also mixing

Theorem (Bowen 1972, 1976)

1. *Axiom A flows are typically mixing. Suffices to have two periodic orbits with periods τ_1, τ_2 and τ_1/τ_2 irrational*
2. *Anosov flows on infranils (tori) are stably mixing*

Theorem (Field, M & Török *Annals of Math 2007*)

Axiom A flows are typically stably mixing

Decay of correlations

Definition

Correlation function $\rho(t) = \int_M v\, w \circ f_t - \int_M v \int_M w$

Mixing $\iff$ $\rho(t) \to 0$ for all v, w with $\int v^2 < \infty$, $\int w^2 < \infty$

Definition

f_t is rapid mixing if $\rho(t) = O(1/t^q)$ for all sufficiently smooth v, w (q arbitrarily large). Also called superpolynomial decay

Theorem (Dolgopyat 1998)

Axiom A flows are typically rapid mixing. Suffices that there are two periods τ_1, τ_2 such that τ_1/τ_2 is diophantine

Theorem (Field, M & Török 2007)

Axiom A flows are typically stably rapid mixing

Rapid Mixing for Nonuniformly Hyperbolic Flows

Theorem (M 2007)

Nonuniformly hyperbolic flows (in sense of Young) are typically rapid mixing

- Lorentz gases (subsequently improved by Chernov (2008) to stretched exponential decay)
- Lorentz gases with external forcing
- Suspended Hénon attractors – flows with quadratic homoclinic tangencies

Also applies to compact group extensions, and implies WIP
Relevant for hypermeander

Mixing for Singular Hyperbolic Flows

Example (Lorenz equations)

$$\begin{aligned}\dot{x} &= 10(y - z) \\ \dot{y} &= 28x - y - xz \\ \dot{z} &= xy - \tfrac{8}{3}z\end{aligned}$$

- Tucker (1999, 2002) The Lorenz attractor exists
- Luzzatto, M & Paccaut (2005) The Lorenz attractor is stably mixing
- Geometric Lorenz attractors are typically stably rapid mixing, Holland & M

Exponential Decay

- Dolgopyat (*Annals of Maths 1998)*, Liverani (*Annals of Maths 2004)* have results on exponential mixing for flows
- Challenge: Find a nontrivial flow that is stably exponentially mixing
- Message: Avoid assumptions on decay of correlations for flows whenever possible!

0-1 Test for Chaos

Gottwald & Melbourne (2004)

- Deterministic time series data $\phi(1), \phi(2), \ldots$
- Choose $c \in (0, \pi)$
- Define $p(n) = \sum_{j=1}^{n} \phi(j) \cos jc$, $q(n) = \sum_{j=1}^{n} \phi(j) \sin jc$
- Define mean square displacement

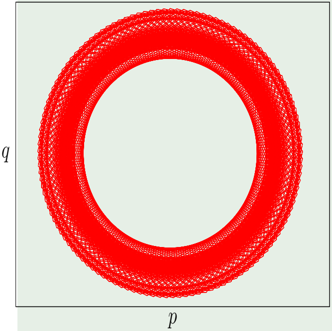
$$M(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} (p(j+n) - p(j))^2$$

- Compute growth rate $K = \lim_{n\to\infty} \log M(n) / \log n$

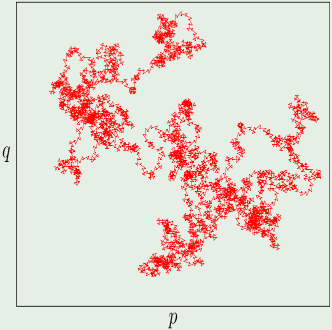
Regular dynamics	Chaotic dynamics
K=0	K=1

Regular Dynamics vs Chaotic Dynamics

Example (Logistic map $f(x) = \mu x(1-x)$, 5000 iterates)



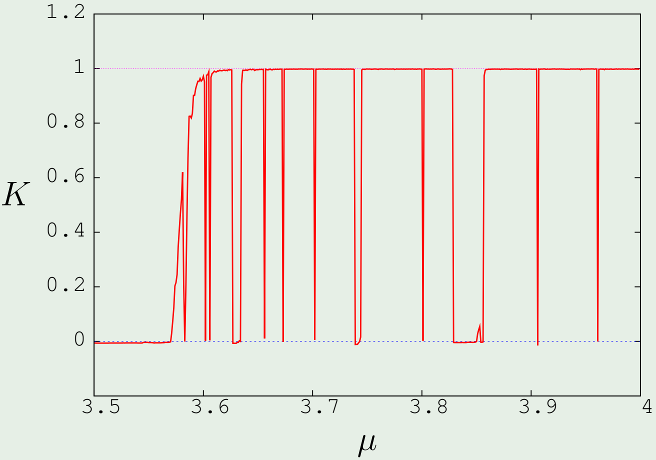
Regular dynamics at $\mu = 3.55$



Chaotic dynamics at $\mu = 3.90$

0-1 Test for Logistic Map

Example (Logistic map $f(x) = \mu x(1 - x)$, 2000 iterates)



Increments of 0.001 for μ

Convergence of Fast-Slow ODEs to SDEs

Melbourne & Stuart

$$\dot{x} = \epsilon^{-1} h_0(y) + f(x,y), \quad x(0) = \xi$$
$$\dot{y} = \epsilon^{-2} g(y), \quad y(0) = \eta$$

$x \in \mathbb{R}^d$: slow variables. $y \in \Lambda$: fast variables
y dynamics governed by chaotic attractor (Λ, μ)

- Assume $\int_\Lambda h_0 \, d\mu = 0$. Set $\int_\Lambda f(x,y) \, d\mu = F(x)$
- Consider stochastic differential equation

$$\dot{X} = \dot{W} + F(X), \quad X(0) = \xi$$

Question: When do solutions for $\dot{x}$ converge to solutions for $\dot{X}$ as $\epsilon \to 0$

Convergence to SDEs

Question: When do solutions for $\dot{x}$ converge to solutions for $\dot{X}$ as $\epsilon \to 0$
Do not assume decay of correlations!

Theorem (M & Stuart)

It is enough to prove weak invariance principle and large deviations for fast dynamics on Λ

Theorem (M & Nicol)

If Λ has a Poincaré map modelled by a Young tower, then Λ satisfies weak invariance principle and large deviations

For example, can take fast equations to be Lorenz equations

More Convergence to SDEs

$$\dot{x} = \epsilon^{-1} h_0(y) + f(x, y), \quad x(0) = \xi$$
$$\dot{y} = \epsilon^{-2} g(x, y), \quad y(0) = \eta$$

More Convergence to SDEs

$$\dot{x} = \epsilon^{-1} h_0(x, y) + f(x, y), \quad x(0) = \xi$$
$$\dot{y} = \epsilon^{-2} g(y), \quad y(0) = \eta$$

Leads to questions about interpretation of stochastic integrals

Example

$d = 1,\ h_0(x, y) = h_1(x) h_2(y)$

$$\dot{X} = h_1(X)\dot{W} + F(X), \quad X(0) = \xi$$

where $\int h_1\, dW$ is Stratonovich

Open problems

- Exponential decay of correlations for (Axiom A) flows
- Decay of correlations for Lorenz attractor
- Good results on ASIP for partially hyperbolic systems
- At least, prove ASIP for equivariant observables of compact group extensions – hypermeander
- Robust counter-examples to 0-1 test?
- More complicated fast-slow systems, interpretation of stochastic integrals