RENORMALIZATION AND SCALING IN APPLIED MATHEMATICS

GUNDUZ CAGINALP
caginalp@pitt.edu

INTRODUCTION. This tutorial is based upon lectures that were given in Bonn-Rottgen, Germany during August 2004. Approximately thirty participants attended this Summer School that was made possible by a grant of the German Research Foundation (DFG) entitled: Priority Program 1095 "Analysis, Modeling and Simulation of Multiscale Problems," and organized by Dr. Christof Eck and Prof. Heike Emmerich.

The lectures are intended to be self-contained, and no special prerequisite material should be necessary in order to understand the material. There are several exercises with solutions that can be used to test one's understanding of the material.

The tutorial is divided into four parts. The first lecture introduces the idea of renormalization group in mathematical structures (e.g., Cantor sets, Koch trianges) that exhibit exact self-similarity. This approach – rather than the historical order in which the research evolved -- facilitates an understanding of the concepts particularly for a mathematical audience. The second lecture focuses on the problem of percolation which can be stated (though not solved) easily in mathematical terms. The problem is that each lattice (e.g., square lattice in two-dimensional space) is occupied with a probability, p, and unoccupied with probability 1-p. The issue is to determine the critical probability at which clusters of occupied lattice points become infinite in size, and to determine the exponents that govern the divergence of related quantities, such as size. These two lectures largely follow the approach of the text by Creswick, Farach and Poole, "Introduction to Renormalization Group Methods in Physics."

Methods under the umbrella of "Renormalization and Scaling (RG)" constitute a common philosophy rather than a specific methodology. In a few mathematical cases, as discussed in Lecture 1, there is complete self-similarity. In other situations, the self-similarity is either approximate or in some asymptotic form. An example of the latter is in the decay, blow-up or finite time extinction of solutions to nonlinear differential equations. In the third lecture, the simplest of these is discussed: a heat equation with a nonlinear source term. The methodology involves a union of classical asymptotic analysis and a systematic approach to RG. The basic work is in (G Caginalp: Phys. Rev. *E. 53*, 66-73,1996), while more recent research has been published in (G Caginalp and H. Merdan: *Discrete and Continuous Systems: Series B*, 3, 565 – 588, 2003).

Finally, in the fourth lecture, an application is made to interface problems. The issue of initial instability has been studied extensively for such problems. Yet there are few results or methods, except in specific exact solutions, for the situation for large time. This lecture demonstrates how one can obtain the key exponent associated with the basic length scale of the interface, namely, $R(t) \sim t^b$, as a function of time. This lecture is based mainly on the following papers:
G Caginalp: *SIAM Applied Mathematics 62*, 424-432 (2001);
G Caginalp and H. Merdan: Physica D, 198, 136-147 (2004)
Nonlinear Analysis (to appear).

Exercises and solutions appear at the end of the lecture material.

SIAM TUTORIAL – RENORMALIZATION AND SCALING METHODS IN APPLIED MATHEMATICS

G. CAGINALP –UNIVERSITY OF PITTSBURGH

LECTURE 1. SELF-SIMILARITY, CANTOR SETS AND FRACTALS, RANDOM WALK – Lectures 1 and 2 based upon Creswick text.

Fractal Dimension and RG. We want to compute the fractal dimension of geometrical objects including random walk. A simple definition (that becomes awkward to use in more complicated situations) is as follows. Since this is mainly motivational material we will not go deeply into this subject and Hausdorff dimension definitions.

Definition. Cover the object with d –dimensional balls of radius $a,$ and let $N(a)$ be the minimum number of balls needed to cover the object. Then the fractal dimension, $D,$ is defined by

$$N(a) \sim a^{-D} \qquad a \to 0.$$

The topological dimension of the object, denoted $d_T,$ is our usual definition. For example, a curve in space has $d_T = 1,$ etc. The overall spatial dimension is $d.$ Then the fractal dimension is

$$d_T \le D \le d.$$

For example, the fractal dimension of a unit square in $\mathbf{R}^2$ is $D = 2.$ Note that in order to obtain a meaningful fractal dimension of the object, it needs to be bounded. For example, if we tried to apply the definition above to the plane, $\mathbf{R}^2,$ we would obtain $N(a) \sim a^{-D} \sim \infty$ for each value of $a.$ Consequently, in problems involving a dynamical component, in which the object is becoming larger in time, we need to rescale it appropriately prior to determining its fractal dimension.

RG Approach to Cantor Sets. Suppose that we remove the middle third of the line $[0, a_o],$ then remove the middle thirds of that, etc. If we continue this process indefinitely we obtain the Cantor set (See Figure 1.1). At the n^{th} stage we have 2^n line segments each of length $a := 3^{-n} a_o.$ This is equivalent to

$$n = -\frac{\ln(a/a_o)}{\ln 3}.$$

It is thereby clear that we can cover the Cantor set by 2^n (one dimensional) balls of radius $a := 3^{-n} a_o,$ i.e.,

$$N(a) = 2^n.$$

And upon substituting the above expression for n we have:

$$N(a) = 2^{-\frac{\ln(a/a_o)}{\ln 3}} = \left(\frac{a}{a_o}\right)^{-(\ln 2/\ln 3)}.$$

When combined with the definition $N(a) \sim a^{-D}$ this relation implies the fractal dimension

$$D = \frac{\ln 2}{\ln 3} \simeq 0.63$$

We want to show that this calculation can be done from a renormalization and scaling type approach. The key idea is to write a relation that transforms one level into the next. In other words, what is the relation between the number of balls needed to cover by $3a$ compared to just a ? If we need $N(a)$ to cover with a balls, then we can go to the next (i.e., coarser) level where the line segments are of size $3a$. But we need only half as many at this level, since the four segments covered by only two at the higher level (as the size increases by a factor of 3). We can express this as a relation of the form:

$$N(3a) = \frac{1}{2} N(a).$$

With the definition $N(a) \sim a^{-D}$ this relation implies:

$$(3a)^{-D} = \frac{1}{2} a^{-D}$$

forcing the relation, once again,

$$3^{-D} = \frac{1}{2} \quad \text{or,} \quad D = \frac{\ln 2}{\ln 3}.$$

In this simple case the latter computation was only slightly easier than the original direct calculation. However, it illustrates an important concept that makes tremendous simplifications in complex situations. In the latter calculation, we simply related the number of balls needed at one level to the next. In other words, if we are able to understand the transformation between the number of balls needed with one size of ball and the the number needed with another, then we should be able to calculate the number, namely $N(a)$, completely. This is basically the idea of coarse graining plus rescaling – the centerpiece of renormalization. For the Cantor sets and other fractal shapes, these relations are exact. In many applications this transformation is either a physical ansatz, a stochastic approximation or an asymptotically valid relation.

FRACTAL PATTERNS. The Cantor set is a one-dimensional example of an exactly self-similar set. Similar methods can be used in higher dimensions where the transformation used above facilitates calculation of the fractal dimension. A simple example is the Koch triangle in which each side of the original equilateral triangle is broken into three pieces. Another equilateral triangle is formed with the middle triangle, etc. (see Figure 1.2). To calculate the fractal dimension, we need only observe the relationships in covering the object (i.e., the shape formed by the successive splitting of the sides) with balls of different radii. In particular, if one can cover the object with $N(a)$ balls of radius a, then it will only take 1/4 as many if we use balls of size $3a$. This is because for each side, there are four equal segments that would be covered with one ball. In other words one has the relation,

$$N(3a) = \frac{1}{4}N(a).$$

As in the Cantor set example, one simply uses the definition, $N(a) \sim a^{-D}$ in order to compute from this relation,

$$(3a)^{-D} = \frac{1}{4}a^{-D}$$

$$4 = a^{-D}/(3a)^{-D} = 3^{D}$$

$$D = \frac{\ln 4}{\ln 3}.$$

Thus all that is needed is the transformation from one size of ball to another. The scaling relation (and fractal dimension) follow as a consequence.

RANDOM WALK -Basic. Consider a random walk in Euclidean dimension $d \geq 2$ consisting of statistically independent steps $\vec{r}$ of fixed length a_o. We are interested in the expected value of the distance from the starting point, which we can assume without loss of generality is the origin. The position after M steps is

$$\vec{R} = \sum_{i=1}^{M} \vec{r}_i.$$

We assume that the direction of the steps is uniformly random, i.e., there is no preferred direction. After M steps the mean displacement is $\langle \vec{R} \rangle = 0$ due to the symmetry assumption. Each step is statistically independent of every other step. We can write this assumption as $\langle \vec{r}_i \cdot \vec{r}_j \rangle = 0$ for each $i \neq j$. A standard probabilistic calculation is

$$\langle |\vec{R}|^2 \rangle = \sum_{i=1}^{M}\sum_{j=1}^{M}\langle \vec{r}_i \cdot \vec{r}_j \rangle = \sum_{i=1}^{M}\langle |\vec{r}_i|^2 \rangle = Ma_0^2.$$

This leads to the Root Mean Square displacement (RMS)

$$\sqrt{\left\langle \left|\vec{R}\right|^2 \right\rangle} = \sqrt{M}\, a_0.$$

Note that this is a basic length scale for the problem. After M steps, the random walker is a distance $\sqrt{M}\, a_0$ (on average) from the starting point. Now, we suppose that M is very large and we split up the random walk into M/n subwalks, each of n steps. We assume that n is also very large, so that $1 << n << M$.

We can apply the preceding RMS analysis to each of these subwalks with n steps, i.e., if $R(n)$ is RMS for n steps, we have

$$R(n, a_0) = \sqrt{n}\, a_0.$$

If we imagine that we ”coarse-grain” the random walk by considering each n steps as a single large step, we need to rescale the step size to $\sqrt{n}\, a_0$. In other words, the RMS analysis suggests that we can consider the random walk of M steps as a random walk of M/n steps of size $\sqrt{n}\, a_0$. In other words, we have $R(M, a_0) \sim \sqrt{M}\, a_0$ and $R(M/n, \sqrt{n}\, a_0) \sim \sqrt{M/n}\, \sqrt{n}\, a_0$.

We can now rescale the lengths in the problem in order to have a meaningful object whose fractal dimension we can calculate. In particular we rescale by root mean square distance we have calculated for the original walk, namely, $a_0\sqrt{M}$, and let $R'(M, a_0/\sqrt{M})$ denote the root mean square distance in terms of the new units. Note that the step size is a length scale that is also rescaled. Evidently, $R'(M, 1/\sqrt{M}) = 1$ and

$$R'(M/n, \sqrt{n}/\sqrt{M}) \sim \sqrt{M/n}\, \sqrt{n}/\sqrt{M} = 1.$$

Now let $N(a)$ be the number of balls of radius a needed to cover the rescaled random walk. If we take the $a := 1/\sqrt{M}$ then $N(1/\sqrt{M}) = M$. However, if we take $\tilde{a} := \sqrt{n}/\sqrt{M}$ then we see that $N(\sqrt{n}/\sqrt{M}) \sim M/n$. That is, we can cover the walk by M balls of radius $1/\sqrt{M}$, or by M/n balls of radius $\sqrt{n}/\sqrt{M}$. These two relations together, namely,

$$N(1/\sqrt{M}) = M. \text{ and } N(\sqrt{n}/\sqrt{M}) \sim M/n.$$

suggest (by simply equating M in these relations) that

$$N(1/\sqrt{M}) = nN(\sqrt{n}/\sqrt{M}).$$

Now using our basic relation, $N(a) \sim a^{-D}$, this relation implies,

$$(1/\sqrt{M})^{-D} = n(\sqrt{n}/\sqrt{M})^{-D}$$

which forces $D = 2$.

In other words, the fractal dimension of this random walk in any spatial dimension $d \geq 2$ is $D = 2$. This means that in two dimensional space, we have $D = d = 2$, so that the random walk in two dimensions covers space. In $d \geq 3$ the random walk does not "cover" space.

An Alternative Look. Again letting $R(M;s)$ be the RMS distance after M steps with step size s, we note that s is the only length scale in the problem. Consequently, R and s must have a linear relationship, while R may depend on M as an arbitrary power (or logarithmic), yielding the form:

$$R(M;s) \sim sM^{\nu}.$$

We noted above that the M step walk of size s could be coarse grained into one with M/n steps of size $\sqrt{n}\,s$. The same scaling relation must hold, so that we can rewrite the above expression as

$$R(M/n;\sqrt{n}\,s) \sim \sqrt{n}\,s(M/n)^{\nu}.$$

Now the key link between these two expressions is that the two RMS distances must be equal (i.e., M small steps yields the same distance as M/n larger steps)

$$R(M/n;\sqrt{n}\,s) = R(M;s)$$

or, using the above expressions,

$$\sqrt{n}\,s(M/n)^{\nu} = sM^{\nu}$$

impying that ν must be 1/2. This means that the full scaling relation is

$$R(M;s) \sim sM^{1/2}.$$

Note the relationship between ν and D, namely, $\nu = 1/D$.

Random Walk with Gaussian probability distribution. In the simplest case we considered, the length of the step was fixed at some $a_o \in \mathbb{R}^{+}$. We now consider the situation in which the vector for each step is now assumed to be given by the distribution

$$p(\vec{r}) = (2\pi\sigma_0^2)^{-d/2} \exp\left(-\frac{|\vec{r}|^2}{2\sigma_0^2} \right).$$

Hence, the probability depends upon the magnitude, vanishing as the magnitude approaches to infinity, and is again symmetric. We are ultimately interested in obtaining a relation of the form (as in the fixed step size example)

$$R(M;\sigma)\sim\sigma M^{\nu}$$

for some ν to be determined, for large M. We coarse grain by considering a new probability distribution (we will use capital with overbar, $\overline{P}$) as a function of $\vec{r}'$ which will be the new random variable that is related to the sum of n original steps,

$$\overline{P}(\vec{r}') = \overline{P}\left(\sum_{i=1}^{n}\vec{r}_i\right).$$

Thus, $\vec{r}'$ performs the function of coarse graining by allowing us to focus on the sum of the "subwalk" of n steps. The new probability distribution can be written in terms of the Dirac delta function, $\delta(\vec{r})$ as,

$$\overline{P}(\vec{r}') = \int_{\mathrm{R}^d}..\int_{\mathrm{R}^d} d^d r_1 \ldots d^d r_n \delta\left(r' - \sum_{i=1}^{n} r_j\right) p(r_1)\ldots p(r_n)$$

where we omit the vector arrow on each of the step vectors, $\vec{r}_1, \vec{r}_2$, etc. on the right hand side and later formulae. By performing the integrals (see Exercise) we can show that this simplifies to the expression:

$$\overline{P}(r') = \frac{1}{(2\pi n\sigma_o^2)^{\frac{d}{2}}}\exp\left(-\frac{|r'|^2}{2n\sigma_o^2}\right)$$

(note factor of 2 typo in text).

We now observe that this expression is identical to that of the original $p(r)$, if we were to rescale the length that governs the step size, namely, $\sigma_{0.}$ The new Gaussian width would be given by

$$\sigma' := \sqrt{n}\,\sigma_o.$$

From this relation we could go directly to the implications of the scaling form $R(M;\sigma)\sim\sigma M^{\nu}$ and assert (as in the fixed step size) that the RMS distance must be the same in either case (original or the smaller number of steps with larger Gaussian width). This would mean that

$$R(M;\sigma_0) = R(M/n;\sqrt{n}\,\sigma_0),$$

or,

$$\sigma_0 M^{\nu} = \sqrt{n}\,\sigma_0 (M/n)^{\nu}$$

from which we conclude that $\nu = 1/2$.

Alternatively we could rescale back the step vectors with the aim of leaving the Gaussian width and invariant. The unique way to accomplish this is by defining a new variable, (omit vector arrows)

$$r'' = \frac{1}{\sqrt{n}} r'$$

so that the new probability distribution (capital P with no overbar) can be written in this new variable (which has been coarse grained and then rescaled, i.e., $r \to r' \to r''$). Noting that

$$d^d r'' = n^{-d/2} d^d r'$$

eliminates the factor of $n^{-D/2}$ outside of the exponent, we have,

$$P(r'') = (2\pi n \sigma_0^2)^{-d/2} \exp\left(-\frac{|r''|^2}{2\sigma_0^2}\right).$$

Clearly this is identical to the original (lower case $p(r)$). Thus if we write the RMS distance in both variables, r and r'', we have

$$R(M;\sigma_o) \sim \sigma_o M^\nu \quad \text{and} \quad R''(M/n;\sigma_o) \sim \sigma_o (M/n)^\nu.$$

Note that the factor M/n in the latter has accounted for the fact that we have fewer steps as a result of the first stage (coarse graining by consolidating n steps into one). Now the only remaining issue is to relate R and R''. The only length scaling (as opposed to coarse graining, or consolidating steps) has been done in the transformation between r' and r''. The RMS lengths correspondingly inherit the scale factor, i.e.,

$$R'' = \frac{1}{\sqrt{n}} R$$

identical to the relation between r'' and r or r'. (Note that there is no scale difference between r and r'; just a summation or consolidation.) Combining this last relation with the scaling above, one obtains

$$\sigma_o (M/n)^\nu = \frac{1}{\sqrt{n}} \sigma_o M^\nu$$

which is the same relation as before, leading to $\nu = 1/2$.

Figure 1.1 Cantor Set

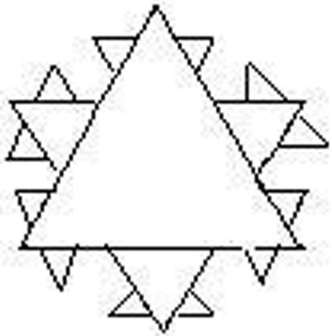

Figure 1.2 Koch Triangles

LECTURE 2. PERCOLATION

1. REVIEW OF RENORMALIZATION AND SCALING IN PHYSICS

Renormalization group theory arose from critical phenomena in physics.

Examples: 1. Liquid-gas at critical point. The equilibrium phase diagram (pressure-temperature) for a typical material shows that the phase boundary separating the liquid and gas terminates at some critical temperature denoted T_c (and pressure).

$$\rho_L - \rho_G \sim (T - T_C)^\beta$$

2. Magnetization. A similar situation occurs for magnetization where one has:

$$M \sim (T - T_C)^\beta$$

Standard statistical mechanics texts, such as the one by Stanley, display typical phase diagrams.

The experimental result is that $\beta \simeq 1/3$ for many materials, although this number is not as universal was originally expected. This means that there will be divergences in the first derivatives of these quantities.

Intriguing aspects: (a) As one moves towards the critical temperature, there is a loss of scale (divergence of basic length scale). (b) ”Universality,” (c) Independent of details of theoretical model

Theoretical modeling: An important general problem of the 20th century (and beyond) has been to understand how an astronomically large number of molecules can interact in a way that leads to the divergence of key thermodynamic quantities. Toward this end, an effort was made to make models that were simple in formulation, but might be adequate to understand the mechanism that led to this behavior.

One such group of models is those of Ising type. Magnetic ”spin” is either up or down. Site is either ”occupied” or not. Each configuration is weighted in accordance with energy level that is determined by the interactions among spins. In the simplest models these interactions are among ”nearest neighbors,” for example, on a square lattice, each spin has four neighbors (in the vertical and horizontal directions).

Calculating the key exponents related to the divergences is extremely difficult even for the simplest models (Onsager 1944, Yang 1952).

The Renormalization methods of Ken Wilson made it possible to calculate these exponents calculations that were tremendously simpler. The basic idea was to perform some type of averaging of blocks of spins. If this is done without some type of ”renormalizing” of the interactions, then one would obtain an obviously incorrect result. Wilson’s key idea was that by understanding the transformation between the original level and the coarse grained one, it would be possible to derive the critical point and the critical exponents.

Rather than discussing these transformations for Ising models, we will focus on a problem that is simpler to state and understand in mathematical terms, without requiring any physics background. This problem of percolation is a very challenging problem in terms of probability theory. In addition to illustrating the basic ideas of renormalization, it illustrates the ease with which important aspects of the problem can be calculated. This material is not rigorous, although there are some symmetry and probability proofs of some of the results. The general problem of proving that these procedures will lead to a convergent sequence (in some class) remains largely open.

2. EXAMPLE OF PERCOLATION

1. Flow of oil through fractured rock
2. Statistical properties of macromolecules
3. Random resistor networks
4. Disordered media

We will consider this problem for a general lattice, and use the example of a square lattice to do explicit calculations.

Basic Problem: Each site of lattice is occupied with probability p and unoccupied with probability $q = 1 - p$. (One can also consider the same situation in terms of bonds between neighbors instead of sites.)

A connected cluster is a group of occupied sites each of which "neighbors" an occupied site.

Example. For a square lattice, "neighbor" denotes any of the four sites that are next to it vertically or horizontally (but not diagonally).

For small p mainly small isolated clusters. As p increases clusters grow larger.

At some critical p_c clusters merge into infinite clusters. Note that an important difference between this percolation problem and the Ising type models is that the latter have a probabilistic weighting for each configuration of spins, depending on the energy level.

Definition. The correlation length, ξ, is the average size of the clusters. The exponent ν is defined by

$$\xi(p) := | \ p - p_c \ |^{-\nu} \tag{2.1}$$

Key step in RG is "coarse-graining" with some factor $b > 0$.

By coarse graining, we obtain a new occupation probability $\overline{p}$ that is a function of the original p and the coarse graining procedure denoted F, i.e.,

$$\overline{p} = F(p) \tag{2.2}$$

As we write down these general relations, we will apply them for the particular example of the square lattice. Another example is suggested in the exercises.

Example: Coarse Graining for Square Lattice. Use the simple rule: if the cluster spans the 'cell' from left to right then cell maps onto an occupied site; otherwise empty. (See Figure 2.1)

$$\begin{aligned} \overline{p} &= F(p) = 2p^2(1-p)^2 + 4p^3(1-p) + p^4 \\ &= 2p^2 - p^4 \end{aligned} \tag{2.2a}$$

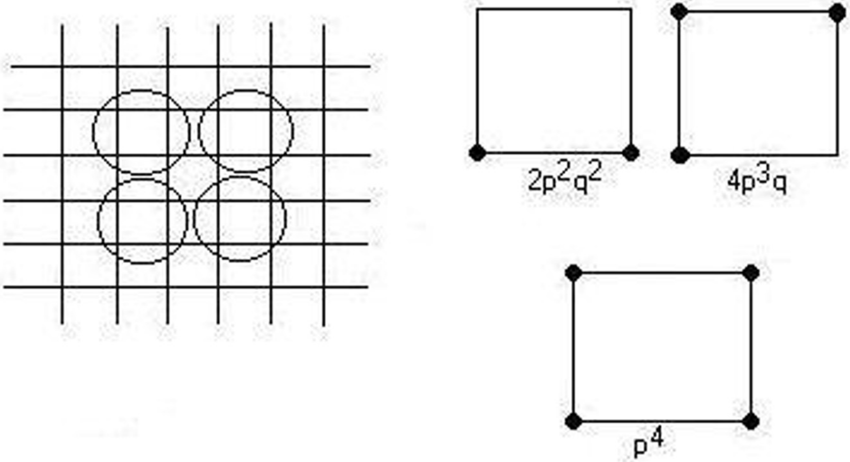

Figure 2.1. The square lattice is subdivided into by a factor of four as shown at left. Coarse graining is accomplished by considering the new lattice sites as occupied if (i) all four sites were occupied, (ii) any three sites are occupied, or (iii) the two on top or on bottom are occupied. Note that (iii) can happen in four ways while (ii) can happen in two ways.

CENTRAL IDEA: The critical probability, p_c, is invariant under this transformation and is a fixed point (along with the trivial fixed points 0 and 1) of the RG transformation:

$$p_c = F(p_c) \tag{2.3}$$

I.e., at the critical probability the system is self-similar. To compute p_c we need only find the solution to the fixed point in (2.2) or (2.2a) above.

Example: Square lattice (2.2a). We need to solve

$$p_c = 2p_c^2 - p_c^4$$

The solutions are : $p_c = 0, p_c = 1, p_c = \frac{1}{2}\sqrt{5} - \frac{1}{2}, p_c = -\frac{1}{2} - \frac{1}{2}\sqrt{5}$. In decimals, $p_c = 0,\ p_c = -1.618,\ p_c = .618,\ p_c = 1.0$.

Note that 0 and 1 are trivial (i.e. lattice is all empty or all occupied) and the negative value is out of the range of probabilities. Hence, the critical probability is

$$p_c = \frac{1}{2}\sqrt{5} - \frac{1}{2} = .618.$$

Note that this is the reciprocal of the golden mean.

COMPUTATION OF EXPONENT ν

When we "average" or translate the original lattice into a coarser grained (by a factor b) lattice, we have:

$$\xi = \text{avg diam cluster} \qquad p = \text{original} \qquad \overline{p} = \text{coarse}$$

$$\xi(p) :=\mid p - p_c \mid^{-\nu}, \qquad \overline{\xi}(\overline{p}) :=\mid \overline{p} - p_c \mid^{-\nu}$$

$$\xi(p) = b\overline{\xi}(\overline{p}) \tag{2.4}$$

$$\text{Implies:} \qquad \mid p - p_c \mid^{-\nu} = b \mid \overline{p} - p_c \mid^{-\nu}$$

Expand analytic function F around p_c so that

$$F(p) = F(p_c) + (p - p_c)\frac{\partial F}{\partial p}(p_c) + \ldots$$

Using (2.2), i.e., $\overline{p} = F(p)$ and (2.3), i.e., $p_c = F(p_c)$ yields

$$\overline{p} = p_c + (p - p_c)\frac{\partial F}{\partial p}(p_c) + \ldots \tag{2.5}$$

Taking (2.5) to the -ν power we have

$$| \overline{p} - p_c |^{-\nu} = | \frac{\partial F}{\partial p}(p_c) |^{-\nu} | p - p_c |^{-\nu}$$

so that using (2.1) and (2.4) yields

$$1 = b | \frac{\partial F}{\partial p}(p_c) |^{-\nu}$$

or,

$$\nu = \frac{\ln b}{\ln F'(p_c)} \tag{2.6}$$

Applying this to our example, we have the following.

Example: For the square lattice, $b = 2$, and, from (2.2a),

$$F'(p) = 4p - 4p^3$$
$$F'(p_c) = 4(.618) - 4(.618)^3 = 1.5279$$

so that (2.6) implies $\nu = \ln 2/\ln(1.5279) = 1.6352$.

The final result for the square lattice problem is:

$$\xi(p) := | p - 0.618 |^{-1.635} \tag{2.7}$$

This calculation can be done with larger b so that a more accurate result is obtained with a bit more combinatorics and algebra suggesting that the true values are close to $p_c = 0.593$ and $\nu = 1.33$.

SIAM TUTORIAL LECTURE 3 – DECAY OF SOLUTIONS TO NAON-LINEAR PARABOLIC EQUATIONS – Based upon article that appeared in Physical Review E 53, 66-73 January 1996

A RENORMALIZATION GROUP CALCULATION OF ANOMALOUS EXPONENTS FOR NONLINEAR DIFFUSION

In this lecture we consider the heat equation with a broad class of nonlinearities and calculate a set of anomalous exponents. Some of the steps in the asymptotic analysis will be left as exercises in the section called "Decay Exerices."

The class of equations we consider is of the form

$$u_t = u_{xx} + f(x, u, u_x, u_{xx}) \tag{1.1}$$

where f is a polynomial of its arguments and is restricted by dimensional analysis considerations as discussed in Section 2. We conclude that the long time and large scale evolution of equation (1.1) with a thin Gaussian of characteristic length l as an initial condition is governed by

$$u(x,t) \sim t^{-(1/2+\alpha)} u^*(x/t^2, 1), \tag{1.2}$$

where $\alpha = (-1)^{p+1} 1\cdot 3\cdot 5 \cdots (p-1)\epsilon B$ for the nonlinearity $\epsilon B x^m u^n u_x^p$ in f. A somewhat more complicated formula is derived for nonlinear terms involving u_{xx} terms. The nonlinearities we consider are those which arise without the introduction of other dimensional parameters, so that the exponent p characterizes the nonlinearity. As noted in GMOL, standard dimensional analysis cannot be used to calculate the exponent α. In many cases such exponents arise as a result of a limit of vanishing length (or other) scale that is a singular rather than a regular perturbation (as discussed in Barenblatt [11]). In this case the dimensionless small number ϵ appears to provide the correction to the classical exponent.

Equations of the form of (1.1) arise in a broad range of diffusion problems in which the detailed physics is taken into account (see for example Ozisik [12], Gebhart [13]). Prigogine [14], pp. 55-68, discusses the limitations of the linear theory of diffusion and derives a number of the key nonlinearities of the form of (1.1) from basic thermodynamics. From a macroscopic perspective, a basic source of nonlinearities involves inhomogeneities in the diffusion coefficient in the flux (Shewmon [15] p.6) or variable dependent potentials [15] p.25 in Fick's laws. Particular examples involve (i) temperature dependent heat conduction, (ii) compressible fluid flow equations McComb [16], (iii) phase transitions involving alloys [17], (iv) seepage flow in which the permeability is dependent on the absolute value of the flow velocity Muscat [18], (v) magnetic fields with permeability depending upon field strength Jackson [19], Ch. 6, (vi) heat diffusion and phase transition problems in which $(temperature)^{-1}$ dependence is considered [14,20], and many other applications [21].

The methodology presented in this paper is useful not only for the exact calculation of large time profiles, but also in establishing equivalence classes in nonlinearities, since the exponents are determined by a simple formula. This also makes possible additional criteria for deciding on models that agree with experiment.

Our analysis also establishes a close link between the large time asymptotics and the blow-up problems (in which u diverges) by using renormalization group methodology. A classical blow-up problem that has been studied extensively is equation (1.1) with f defined as u^n for $n > 1$. Berger and Kohn [22] have used rescaling arguments as part of a numerical scheme to determine the singularity. Recently, Bricmont and Kupiainen [10] have provided rigorous proofs of the existence of infinitely many profiles around the blow-up point.

2. ASYMPTOTICS OF THE HEAT EQUATION WITH SMALL NONLINEARITY. Let ϵ be a small dimensionless number. We consider the heat equation with nonlinearity in the form

$$C_p u_{t'} = K\{u_{xx} + \epsilon N[x, u, u_x, u_{xx}]\} \tag{2.1}$$

where C_p and K are constants (with $D := K/C_p$) and N is a sum of terms of its arguments of the form

$$x^m u^n u_x^p u_{xx}^q \tag{2.2}$$

where m, n, p, q are integers that satisfy

$$n + p + q = 1 \qquad \textit{and} \qquad p + 2q - m = 2, \qquad (2.2a, b)$$

so that (2.1) is dimensionally correct without the introduction of a new time or space scale. We also assume that u is dimensional (e.g. temperature) so that (2.2a) is required.

Examples: (i) $u^{-1}u_x^2$, (ii) $x^{-1}u_x$, (iii) $xu^{-2}u_x^3$, (iv) $x^2u^{-2}u_x u_{xx}^2$.

Note that (i) arises from the inverse temperature diffusion,

$$(u^{-1})_{t'} = D(u^{-1})_{xx} \qquad \textit{or,} \qquad u_{t'} = D(u_{xx} - 2u_x^2/u). \tag{2.3}$$

In general, diffusion processes obtained in the manner of (2.3) that do not involve additional physical constants will lead to these types of nonlinearities.

To simplify notation and maintain correspondence with GMOL we define $t := 2Dt'$ (units $length^2$) and use (2.1) in the form

$$u_t = \frac{1}{2} u_{xx} + \epsilon N[x, u, u_x, u_{xx}] \tag{2.1$'$}$$

subject to the initial condition

$$u(x; l) := g(x; l) := \frac{Q_0}{(2\pi l^2)^{1/2}} exp\left(\frac{-x^2}{2l^2}\right) \tag{2.4}$$

where $Q_0 := T_0 Q_1$ with T_0 having temperature units and Q_1 length units. We will be interested in a sharply peaked Gaussian so that l will be small. One of the subtleties in the asymptotics, however, is that the scale of l compared with ϵ Using the Green's function

$$G(x, t) := (2\pi t)^{-1/2} exp\left(\frac{-x^2}{2t}\right) \tag{2.5}$$

and treating the nonlinearity as a source term we can express the solution to (2.1') as

$$u(x, t) = \int_{-\infty}^{\infty} dy G(x-y, t) g(y) + \epsilon \int_0^t ds \int_{-\infty}^{\infty} dy G(x-y, t-s) N[y, u(y, s), ...]. \tag{2.6}$$

We solve this using an asymptotic expansion for small ϵ and write the formal sum,

$$u(x, t; \epsilon, l) = u_0(x, t; l) + \epsilon u_1(x, t; l) + ... \tag{2.7}$$

so that l is not yet treated as a small number in comparison with ϵ. Formally solving (2.6) by substituting (2.7) and retaining only $O(1)$ terms leads to the expression (see Exercise)

$$u_0(x, t) = \frac{Q_0}{[2\pi(t + l^2)]^{1/2}} exp\left(\frac{-x^2}{2(t + l^2)}\right) \tag{2.8}$$

where we have suppressed the parameter l. The derivatives of u_0 are given by

$$\frac{\partial u_0}{\partial x} = \left(\frac{-x}{t+l^2}\right) u_0 \qquad \textit{and} \qquad \frac{\partial^2 u_0}{\partial x^2} = (t+l^2)^{-1}\left(\frac{x^2}{t+l^2} - 1\right) u_0 \quad (2.9)$$

so that one has the relation

$$\frac{\partial^2 u_0}{\partial x^2} = -(t+l^2)^{-1}\left(u_0 + x\frac{\partial u_0}{\partial x}\right). \qquad (2.10)$$

The last identity will establish a simple relation between the effects of nonlinearities involving the second order derivatives and the lower order in terms of the anomalous exponents. We proceed by using u_0 in the expression (2.6) to generate the next term of (2.7), namely u_1. Since N consists of a linear sum of terms of the form (2.2), it suffices to consider the nonlinear term $x^m u^n u_x^p u_{xx}^q$ subject to (2.2). For convenience, we consider the case $q = 0$ first, so that the nonlinearity will be completely specified by p, as $n = 1 - p$ and $m = p - 2$. We then have (see Exercise)

$$u_1(x,t;l) = \int_0^t ds \int_{-\infty}^{\infty} dy \frac{(t-s)^{-1/2}}{(2\pi)^{1/2}} exp\left(\frac{-(x-y)^2}{2(t-s)}\right) \frac{Q_0}{(2\pi)^{1/2}} y^m$$

$$\cdot (s+l^2)^{-1/2} \frac{(-y)^p}{(s+l^2)^p} exp\left(\frac{-y^2}{2(s+l^2)}\right)$$

$$\cong \frac{Q_0}{2\pi} t^{-1/2} e^{-x^2/2t} \int_0^t ds (s+l^2)^{-p-1/2} (-1)^p \int_{-\infty}^{\infty} dy y^{p+m} exp\left(\frac{-y^2}{2(s+l^2)}\right)$$
$$+ \quad terms \quad that \quad are \quad smaller \quad in \quad l. \tag{2.11}$$

The approximations involve replacing $t-s$ by t, and $x-y$ by x to obtain the $t^{-1/2}e^{-x^2/2t}$ term. The justification, which is discussed in detail in Section 4, is based on the Laplace's method for integrals (Erdelyi [23], p. 36), since the main contribution to the integral must arise from the regions near $y=0$ and $s=0$ for small l. Note that the entire term is of the same order in ϵ so that the smaller terms in l cannot cancel the higher order. Let I_1 denote the integrand of the s integral and use $w := y/(s+l^2)^{1/2}$ to obtain (for $p \geq 1$)

$$I_1 = (s+l^2)^{-1}(-1)^p \int_{-\infty}^{\infty} w^{2(p-1)} e^{-w^2/2} dw$$

$$= \frac{(-1)^p (2\pi)^{1/2}}{(s+l^2)} \{1 \cdot 3 \cdot \quad |2(p-1)-1|\}. \tag{2.12}$$

Combining this with (2.8) one has to leading order in ϵ and to leading order in l within $O(\epsilon)$ the solution

$$u(x,t;\epsilon,l) = \frac{Q_0 t^{-1/2}}{(2\pi)^{1/2}} e^{-x^2/2t} \left\{1 + \epsilon(-1)^p 1 \cdot 3 \cdot \quad \cdot |2p-3| log\left(t/l^2\right) |\right\} \tag{2.13}$$

for $p \geq 1$ and $q := 0$.

Nonlinearities involving u_{xx} :

The nonlinearity $x^m u^n u_x^p u_{xx}^q$ (with $q \neq 0$) leads to (see Exercise)

$$u_1(x,t;l) \cong \frac{Q_0}{2\pi} t^{-1/2} e^{-x^2/2t} \int_0^t ds I_1(s,y;l)$$

$$I_1 := (s+l^2)^{-1}(-1)^p \int_{-\infty}^{\infty} w^{p+m}(w^2-1)^q e^{-w^2/2} dw. \tag{2.14}$$

Using the binomial theorem to evaluate I_1, one obtains for

$$N[x,u,u_x,u_{xx}] = B(m,n,p,q)x^m u^n u_x^p u_{xx}^q$$

the result

$$u(x,t;\epsilon,l) = \frac{Q_0 t^{-1/2}}{(2\pi)^{1/2}} e^{-x^2/2t}. \tag{2.15}$$

$$\cdot \left\{ 1 + \epsilon B(m,n,p,q) \sum_{j=0}^{q} (-1)^{j+p} 1 \cdot 3 \cdot \quad \cdot |2p+4q-2j-3| log\left(t/l^2\right) \right\}.$$

In particular, for the special case in which $m = n = p = 0$ and $q = 1$ one obtains

$$\sum_{j=0}^{1} (-1)^j 1 \cdot 3 \cdot \quad \cdot |4-2j-3| log\left(t/l^2\right) = 0$$

indicating as expected that the addition of a linear term u_{xx} does not make a contribution and will not change the exponent.

Porous medium equation: The porous medium equation considered by GMOL, can be written as

$$u_t - \frac{1}{2}u_{xx} = \frac{\epsilon}{2} H(-u_{xx})u_{xx} \tag{2.16}$$

where $H(z) := 1$ if $z > 0$ and vanishes otherwise. Hence,

$$H(-u_{0yy}) = H\left(1 - \frac{y^2}{s+l^2}\right)$$

and the integral $\int_{-\infty}^{\infty}(w^2-1)e^{-w^2/2}dw = 0$ is truncated to $\int_{-1}^{1}(w^2-1)e^{-w^2/2}dw \neq 0$ so a nontrivial contribution to u_1 via (2.14) is possible. Thus it appears that all but a small fraction of nonlinearities similar to

(2.16) will result in a nonzero contribution that will lead to an anomalous exponent. In other words the integral

$$\int_{-\infty}^{\infty} F\{(w^2-1)e^{-w^2/2}\}(w^2-1)e^{-w^2/2}dw = 0 \qquad (2.17)$$

will vanish for $F(z) := 1$, the linear case, and for a set of functions that represent a particular symmetry that weights the function equally on either side of $w = 1$ with respect to the particular symmetry that originates from the second derivative of the Gaussian.

3. THE RENORMALIZATION GROUP FROM AN ANALYSIS PERSPECTIVE. Given an asymptotic relation such as (2.15) one can calculate the anomalous exponent explicitly and obtain the precise similarity solution for large time and space. The arguments are within the context of formal applied analysis without reference to numerical procedures or physical analogies, and are thus in a form that can provide a basis for attempting rigorous proofs and generalizations to a wide variety of nonlinear differential equations. The methodology is close in spirit to those of Goldenfeld, Martin and Oono [8] and Creswick, Farach and Poole [3]. For the problem under consideration, the result can be stated as follows (using true dimensions):

Proposition 3.1. Suppose u can be expressed as

$$u(x,t';\epsilon,l) = \frac{T_0\left(\frac{t'}{Q_1^2/D}\right)^{-1/2}}{2\pi^{1/2}} e^{-x^2/(4Dt')}\left\{1+\epsilon A log\left(2Dt'/l^2\right)\right\} \qquad (3.1)$$

where A does not depend upon x, t', ϵ or l, and $\epsilon A log\left(2Dt'/l^2\right) << 1$. Then, to leading order in ϵ, u can be written as

$$u(x,t';\epsilon,l) = \left(\frac{t'}{Q_1^2/D}\right)^{-\frac{1}{2}+\epsilon A} u^*\left(\frac{x}{(Dt'/Q_1^2)^{1/2}}, \frac{Q_1^2}{D}\right) \qquad (3.2)$$

so that the anomalous exponent is given by ϵA. The fixed point function u^* is given by

$$u^*(\zeta,\tau_1) = \frac{T_0}{2\pi^{1/2}} exp\left\{-\frac{\zeta^2}{4D\tau_1}\right\}\left\{1+\epsilon A log\left(\frac{2D}{l^2}\tau_1\right)\right\} \qquad (3.3)$$

Verification: The derivation is divided into five stages that can be implemented more generally on other problems.

Stage 1. One needs to obtain an identity [up through $0(\epsilon)$] of the form

$$u(b^{\phi}x, bt') = Z(b)u(x, t') \tag{3.4}$$

that is valid for a particular choice of Z and ϕ and all $b > 1$. For the problem under consideration, clearly the exponential term in (3.1) forces $\phi = 1/2$. Rewriting (3.1) to $O(\epsilon)$ one has

$$u(b^{1/2}x, bt') = \frac{T_0}{(2\pi)^{1/2}} \left(\frac{2t'}{Q_1^2/D}\right)^{-1/2} b^{-1/2} e^{-\frac{x^2}{4Dt'}} \cdot \tag{3.5}$$

$$\cdot \left\{1 + \epsilon A log(t/l^2)\right\} \left\{1 + \epsilon A log b\right\}$$

so that (3.4) is satisfied with $\phi = 1/2$ and

$$Z(b) := b^{-1/2}(1 + \epsilon A log b). \tag{3.6}$$

Note that Z does not depend upon l.

Following Creswick, Farach and Poole [3] we define the operator

$$R_{b,\phi}u(x, t') := \frac{1}{Z(b)} u(b^{1/2}x, bt'). \tag{3.7}$$

Stage 2. By iteration we have (again suppressing ϵ and l and ignoring $O(\epsilon^2)$ terms),

$$u(b^{k/2}x, b^k t') = Z(b)^k u(x, t') \tag{3.8}$$

A fixed point of this iteration will exist only if

$$u^*(x, t') := \lim_{k\to\infty} Z(b)^{-k} u(b^{k/2}x, b^k t') \tag{3.9}$$

is well defined. We assume the existence of a fixed point in this formal derivation and rewrite (3.9) for large but finite k as

$$u(b^{k/2}x, b^k t') \cong Z(b)^k u^*(x, t'). \tag{3.10}$$

Note that $b > 1$ was necessary for considering large time and space, and in fact for the assumption of approximate self-similarity that underlies the existence of the fixed point u^*. We rewrite the last equation by defining

$$\overline{x} := b^{k/2}x \qquad and \qquad \overline{t} := b^k t' \tag{3.11}$$

so that one has (for large k)

$$u(\overline{x},\overline{t}) \cong Z(b)^k u^*(\overline{x}b^{-k/2},\overline{t}b^{-k}). \tag{3.12}$$

This means that for any large $\overline{t}$ we can determine the u profile be setting $b^k := \overline{t}/(Q_1^2/D)$, so that the second argument remains unchanged as we examine different values of $\overline{t}$, and we can then write (3.12) as

$$u(\overline{x},\overline{t}) \cong \left\{Z\left\{\left(\frac{\overline{t}}{Q_1^2/D}\right)^{1/k}\right\}\right\}^k u^*\left(\frac{\overline{x}}{(D\overline{t}/Q_1^2)^{1/2}},\frac{Q_1^2}{D}\right). \tag{3.13}$$

Note that if we chose to ignore the units we would set the second argument of u^* at unity.

Stage 3. The scaling exponent will be determined by the limit

$$\lim_{k\to\infty}\left\{Z\left\{\left(\frac{\overline{t}}{Q_1^2/D}\right)^{1/k}\right\}\right\}^k$$

if it exists. To calculate this we let $t_1 := D\overline{t}/Q_1^2$ and substitute directly into (3.6) and utilize the asymptotic expansion

$$\left(1+\frac{\epsilon A}{k}logt_1\right)^k \cong t_1^{\epsilon A} \tag{3.14}$$

to obtain

$$\left[Z\left(t_1^{1/k}\right)\right]^k = t_1^{-1/2}\left\{1+\frac{\epsilon A}{k}logt_1\right\}^k \cong t_1^{-1/2+\epsilon A}$$

so that

$$\lim_{k\to\infty}\left\{Z\left(\frac{D\overline{t}}{Q_1^2}\right)^{1/k}\right\}^k = \left(D\overline{t}/Q_1^2\right)^{-\frac{1}{2}+\epsilon A}. \tag{3.15}$$

Stage 4. Using (3.15) in (3.13), and dropping the superbar since (3.13) is valid for arbitrary large $\overline{t}$, one obtains

$$u(x,t') = \left(Dt'/Q_1^2\right)^{-\frac{1}{2}+\epsilon A} u^* \left(\frac{x}{(Dt'/Q_1^2)^{1/2}}, \frac{Q_1^2}{D} \right) \tag{3.16}$$

so that the anomalous exponent or "dimension" is $\alpha = -\epsilon A$.

Stage 5. Explicit evaluation of u^* is possible by writing (3.1) as

$$u(x,t';\epsilon,l) = \frac{T_0 \left(\frac{t'}{Q_1^2/D} \right)^{-1/2}}{2\pi^{1/2}} e^{-x^2/(4Dt')} \left\{1 + \epsilon A log\left(Dt'/Q_1^2\right)\right\} \left\{ 1 + \epsilon A log \left(\frac{2Q_1^2}{l^2} \right) \right\}$$

and utilizing (3.14) again to obtain

$$u(x,t';\epsilon,l) = \frac{T_0}{2\pi^{1/2}} \left(\frac{Dt'}{Q_1^2} \right)^{-\frac{1}{2}+\epsilon A} e^{-x^2/4Dt'} \left\{ 1 + \epsilon A log \left(\frac{2Q_1^2}{l^2} \right) \right\}. \tag{3.17}$$

Comparison of (3.17) with (3.16) leads to the evaluation of u^* as

$$u^* \left(\frac{x}{(Dt'/Q_1^2)^{1/2}}, \frac{Q_1^2}{D} \right) = \frac{T_0}{2\pi^{1/2}} exp \left\{ - \frac{\left(\frac{x}{(Dt'/Q_1^2)^{1/2}} \right)^2}{4D(Q_1^2/D)} \right\} \left\{ 1 + \epsilon A log \left(\frac{2Q_1^2}{D} \cdot \frac{D}{l^2} \right) \right\}. \tag{3.18}$$

which is (3.3) so Prop. 3.1 has been verified.

The results of Section 2 then imply the following result.

Proposition 3.2. The general nonlinearity $N[x, u, u_x, u_{xx}]$ defined in (2.15) has the anomalous exponent

$$\alpha = -\epsilon B(m,n,p,q) \sum_{j=0}^{q} (-1)^{j+p} 1 \cdot 3 \cdot \quad \cdot |2p + 4q - 2j - 3|. \tag{3.19}$$

While the calculation has used the idea of small ϵ, the expectation of universality would imply that the exponent varies continuously as ϵ is made larger provided a singularity does not occur due to the nonlinearity.

We skip Section 4 of the paper which is more technical.

5. A RG CALCULATION OF BLOW-UP. The RG analysis of Section 3 extends easily to blow-up of solutions to differential equations. In particular, we consider the equation

$$u_t = u_{xx} + u^r \tag{5.1}$$

with $r > 1$. Note that a time scale, τ, is needed as a coefficient of u^r in (5.1) (provided that u is dimensionless, e.g. $u = concentration$). If u is dimensional, e.g. $u = temperature$, then the coefficient τ^{-1} is replaced by $\tau^{-1}T_0^{-r-1}$. We suppress these units at this stage since the issue is essentially the same as in Section 3. One can regard (5.1) as the natural units in which time is measured by τ and space by $(D\tau)^{1/2}$.

Berger and Kohn [22] and references therein consider the the interval $-1 < x < 1$ with Dirichlet boundary conditions

$$u(-1, t') = u(1, t') = 0. \tag{5.2}$$

and initial data, $\phi(x)$, such that

$$\phi > 0, \quad \phi(x) = \phi(-x), \quad x\phi'(x) < 0 \quad for \quad x \neq 0, \tag{5.3}$$

for which the solution to (5.1), (5.2) is known to be positive, symmetric and decreasing in $|x|$. Furthermore, it is known (Giga and Kohn [24], Galaktionov and Posashkov [25]; see also references within) that the solution to (5.1)-(5.3) exhibits a divergence at some time $t_0 > 0$ so that

$$u(x, t) \sim ((r-1)(t_0 - t))^{\frac{-1}{r-1}} \tag{5.4}$$

is the leading order (in $|t_0 - t|^{-1}$) term in the solution. Our analysis here is local and applies to any set of initial and boundary conditions for which the solution has similar qualitative behavior.

We will derive (5.4) using the methods of Section 3 so that the coefficient, in addition to the exponent of the leading term is obtained through a simple calculation. The arguments are close to those of Berger and Kohn [22], who based a numerical scheme on the identity (5.5) below in order to calculate the singularity.

Geometrically, it is easy to see why the methods of Section 3 should apply. In the asymptotic decay problems, the fixed point of the RG methods amounted to a near self similarity as the solution decayed to zero as t approached ∞. For the blow-up problems there is an analogous situation in that the solution is nearly self similar as t approaches t_0, that is, u approaches ∞. To put it more simply, the picture in the blow-up problems looks like the decay problems upon rotating the graph counterclockwise by $\pi/2$.

To define the proper RG operator $R_{b,\phi}$ one observes as before that only $\phi = 1/2$ will be possible since the leading space derivative is second order, so that scaling time by a factor b forces a scaling of space by $b^{1/2}$. Substitution of $u(b^{1/2}x, bt)$ into the differential equation (5.1) leads immediately to the conclusion that u itself must be scaled by a factor of $b^{1/(\mathrm{r}-1)}$, so that if $u(x, t-t_0)$ solves (5.1) then so does

$$R_{b,1/2}u(x,t-t_0) = u_b(x,t-t_0) = b^{1/(\mathrm{r}-1)}u(b^{1/2}x, b(t-t_0)). \tag{5.5}$$

Defining $Z(b) := b^{1/(\mathrm{r}-1)}$ one can write the transformation in the standard form as in Section 3. A repeated application of this transformation with $b < 1$ will move the solution closer to the singularity and closer to self similarity. In the limit one may expect the existence of a fixed point function u^* satisfying

$$u^*(x,t-t_0) = lim_{k\to\infty} b^{k/(\mathrm{r}-1)}u(b^{k/2}x, b^k(t-t_0)). \tag{5.6}$$

Letting $\overline{x} := b^{k/2}x$ and $\overline{t} := b^k(t-t_0)$ and writing (5.6) as an approximation for large k, one has (to leading order in k^{-1})

$$u(\overline{x},\overline{t}) = b^{\frac{-k}{\mathrm{r}-1}}u^*(b^{-k/2}\overline{x}, b^{-k}\overline{t}) \tag{5.7}$$

For any (small) $\overline{t}$ we can evaluate $u(\overline{x},\overline{t})$ by setting $b^k = -\overline{t}$ so that

$$u(\overline{x},\overline{t}) = (-\overline{t})^{\frac{-1}{\mathrm{r}-1}}u^*(\overline{x}/\sqrt{-\overline{t}}, -1) \tag{5.8}$$

for $\overline{t}$ near 0 and remains valid as an approximate solution to (5.1) when $\overline{t}$ is replaced by $t-t_0$. Hence the exponent $-1/(\mathrm{r}-1)$ is thereby determined.

One can proceed to find the coefficient by substituting this expression into the original differential equation. Upon defining

$$f(\zeta) := u^*(\overline{x}/\sqrt{-\overline{t}}, -1), \qquad \zeta := x/\sqrt{t_0 - t} \tag{5.9}$$

one can write the differential equation to leading order in $|t_0 - t|^{-1}$ as

$$-f''(\zeta) + \frac{\zeta}{2} f'(\zeta) + \frac{1}{\mathrm{r} - 1} f(\zeta) - f(\zeta)^r = 0, \tag{5.10}$$

with solution

$$f(\zeta) = (\mathrm{r} - 1)^{\frac{-1}{r-1}}. \tag{5.11}$$

Substitution of (5.11) into (5.8) implies the formula (5.4), which in more precise form can be written as

$$\lim_{t \nearrow t_0} \frac{u(x,t)}{((r-1)(t_0 - t))^{\frac{-1}{r-1}}} = 1. \tag{5.12}$$

Remark 5.1. Similar conclusions about the nature of blow-up can be drawn from guessing the form

$$u(x,t) = t^{-s} f\{x(t_0 - t)^{-1/2}\}, \tag{5.13}$$

since substitution into the differential equation shows that a solution can be attained with the exponent and functional form of f given by (5.4).

6. CONCLUSION. We have shown that renormalization methods can be used to calculate the anomalous exponent of the heat equation with a class of nonlinearities. Furthermore, this can be done within a systematic applied mathematical setting of asymptotic analysis. The axiomatic set of steps involved in the calculation thus has the potential to be made completely rigorous and does not rely upon physical analogy with other phenomena.

The renormalization group within this setting becomes a simple computational tool that can be generalized to other classes of partial differential equations whenever there is basic solution upon which perturbations (e.g. nonlinearities of order ϵ) can be calculated using the usual asympotic analysis procedures.

The nonlinearities we have considered include the inverse temperature heat equation, $(T^{-1})_t = (T^{-1})_{xx}$, as a special case. This equation along with source terms has been used in some phase transition problems. For temperatures that are far from absolute zero, the difference between this equation and the ordinary heat equation amounts to a nonlinear source term as noted in equation (2.3). The fact that the two equations differ in the large time exponent gives a clear criterion for the use of each in modelling. In fact, this procedure of calculating the large time behavior using these RG methods can be used in conjunction with experiment and statistical mechanics calculations in order to decide on the appropriate equations in a particular problem.

Our analysis also unifies the methodology involved in blow-up problems with those of large time decay. The key first step in both problems is to obtain a transformation that leads to a fixed point, indicating that the solution asymptotically to a self-similar graph. In the case of the large time decay, the solution approaches self-similarity as (u, t) tends to $(0, \infty)$. In the case of blow-up the situation is identical except for a $\pi/2$ rotation so that self-similarity is approached as (u, t) tends to (∞, t_0). The second step in both cases involves extracting functional relationships based on the existence of a fixed point as the transformation is applied a large number, k, times. Using a large but still finite k, one can obtain the unique functional form that is compatible with the fixed point. This procedure is philosophically similar to using repeated rescaling of the numerical grid but allows a direct and simple calculation of the exponent as well as the coefficient of the singularity. Numerical computation can be avoided in the blow-up problems just as appeal to physical analogy can be avoided in the asymptotic decay problems, as the underlying mathematical structure is in fact quite simple.

Throughout this analysis ϵ has been a small parameter. It has been noted by GMOL that the results could be continued for larger ϵ in the spirit of the Wilson-Fisher ϵ-expansion. The formalism of Section 3 may be used in conjunction with analytic continuation methods to prove the validity of the expansion beyond the usual rigorous asymptotics. In most cases asymptotic calculations are made rigorous within an arbitrarily small neighborhood $\epsilon \in (0, \epsilon_0)$. The continuation methods may be useful in extending the arbitrarily small ϵ_0 to a known finite number.

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LECTURE 4. INTERFACE PROBLEMS

Part 1. The fully dynamic (parabolic) case.

For many interface problems we need to know about the large time behavior. Numerical computations and even experiments have difficulty resolving the issues (see Figure 4.1).

KEY QUESTIONS:

Can we determine the characteristic length, $R(t) \sim t^{\beta}$?

The characteristic length could be, for example, the radius of the smallest circle or sphere enclosing the interface (”radius of gyrations”).

What values of β are in the permissible spectrum?

What parameters in the system are ”relevant?”

Early work by Jasnow and Vinals in $d = 2$ indicates $\beta = 1$ in quasi-static one-sided.

Caginalp (SIAM Applied Math 2001) considers fully dynamic case below:

$$
\begin{aligned}
CT_t &= K\triangle T && \text{in } \Omega \\
lv_n &= -K[\nabla T \cdot \hat{n}]_-^+ && \text{on } \Gamma(t) \qquad (4.1) \\
T &= \frac{-\sigma_0}{[s]_{eq}}(\kappa + \alpha v_n) && \text{on } \Gamma(t)
\end{aligned}
$$

Here, C is the specific heat per unit volume, K is the thermal conductivity (and $D := K/C$), l is the latent heat per unit volume, σ_0 is the surface tension, $[s]_{eq}$ is the entropy difference per unit volume between phases, α is the dynamical undercooling and $[\ldots]_-^+$ is the difference in the limiting values between the two sides of the interface. The variables v_n and κ denote the (normal) velocity and the sum of the principle curvatures at a point on the interface, respectively. In addition, + denotes the phase with the higher internal energy, i.e., liquid and – denotes the phase with the lower internal energy, i.e., solid.

We can rewrite (4.1) and (4.2) in the single equation formulation as,

$$CT_t - K\triangle T = -\frac{l}{2}\phi_t$$

so that the latent heat is treated like a source term.

Use the fundamental solution (assume Ω is infinite or very large), G, with $D := K/C$ as,

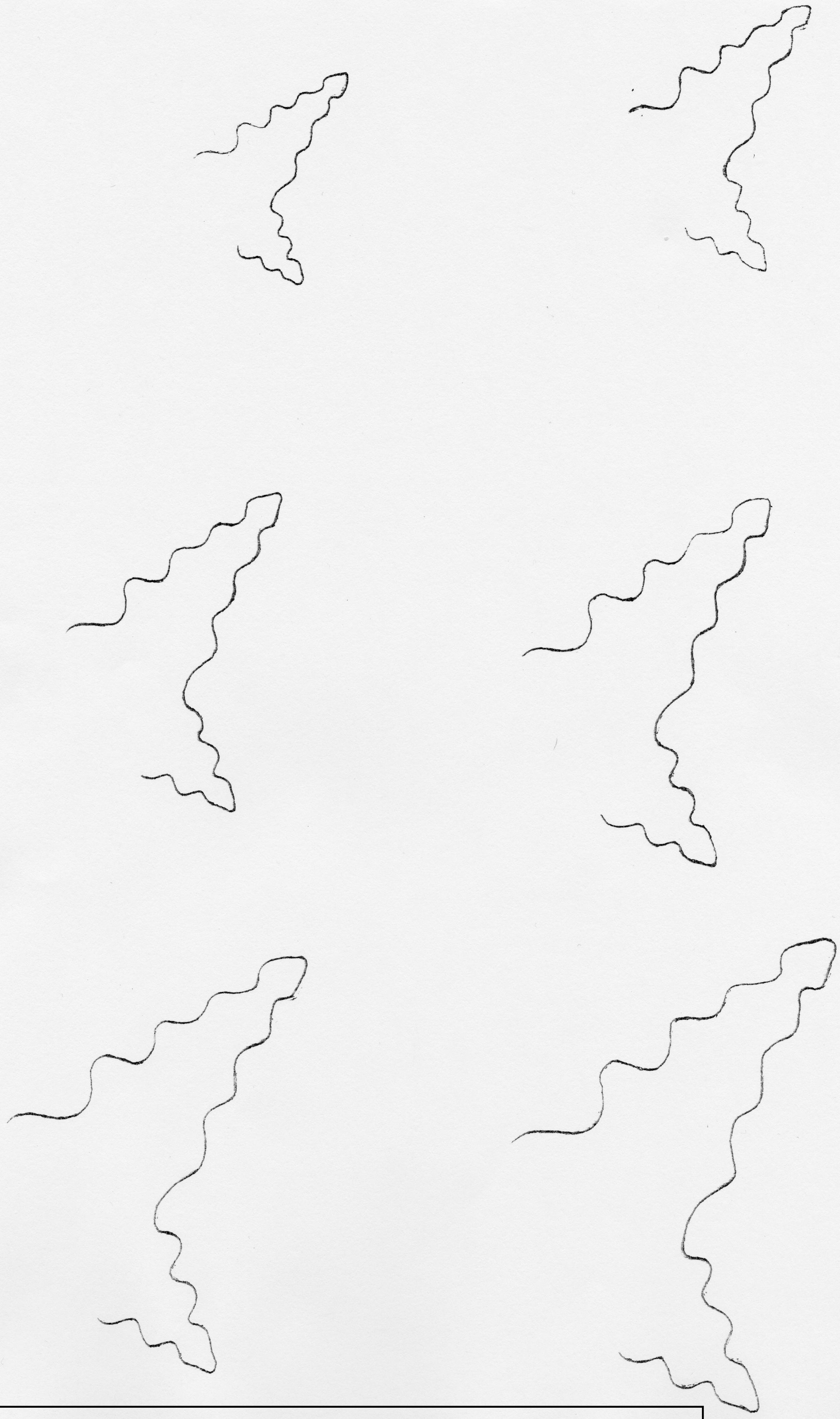

Figure 4.1: A dendrite that is self-similar with a single scale would have the time evoluitions show in the six figures. The interface shape at one time can be used to generate the interface at all later times, once one determines the necessary exponent.

$$G(\vec{y},t) = (4\pi Dt)^{-d/2}\exp\left\{-(4Dt)^{-1} \mid \vec{y} \mid^2 \right\}.$$

The function ϕ_t will vanish outside the interfacial region. Across the interface it will behave like a delta function. We define a local coordinate system $\left(r,\vec{\sigma}\right)$ in a sufficiently narrow region near the interface, so that r is a signed normal distance to the interface (positive toward the liquid) and $\vec{\sigma}$ the tangential coordinates. Using this system, we can write the normal velocity of each point on the interface as

$$v_n(x,t) = -r_t(x,t)$$

we approximate ϕ by a function Φ depending only on r, i.e., $\phi(x,t) \simeq \Phi(r(x,t))$, so that the time derivative is given by

$$\phi_t(x,t) \simeq \Phi_t(r(x,t)) = r_t\Phi_r(r(x,t)) = -v_n\Phi_r(r(x,t))$$

Substituting into the Greens' function equation, we have,

$$\begin{aligned} T(x,t) = \int_0^t ds\int_\Omega d^dyG(x-y,t-s)\,\cdot \\ \left(-\frac{l}{2C}\right)(-v_n(y,s))\,\cdot \\ \Phi_{r_y}(r_y(y,s) + IC. \end{aligned}$$

The derivatives of Φ vanish just outside the interfacial region, we can perform the integral in the normal direction reducing the integral over Ω to one over Γ, with the result (omiting IC),

$$\begin{aligned} T(x,t) = \int_0^t ds\int_{\Gamma(s)} d^{d-1}\sigma_yG(x-y,t-s) \\ \left(\frac{l}{2C}\right)v_n(y,s) \end{aligned}$$

For points (x,t) on the interface, one has:

$$\begin{aligned} & d_0\{\kappa(x,t) + \alpha v_n(x,t)\} \\ = & \int_0^t ds\int_{y\epsilon\Gamma(s)} d^{d-1}\sigma_{y'}G(x-y,t-s)v_n(y,s) \end{aligned} \tag{4.2}$$

We need now to convert to dimensionless counterparts in order to compare pure numbers.

THE RG PROCEDURE

Stage 1 Algebraic sub bx for x , and $b^{-\lambda}t$ for t yields

$$d_0\{\kappa(bx, b^{-\lambda}t) + \alpha v_n(bx, b^{-\lambda}t)\}$$
$$= \int_0^{b^{-\lambda}t} ds \int_{y\epsilon\Gamma(s)} d^{d-1}\sigma_{y'} G(bx - y, b^{-\lambda}t - s) v_n(y, s)$$

Next, define new variables $s' = s/b^{-\lambda}$, $y' = y/b$ so that

$$d_0\{\kappa(bx, b^{-\lambda}t) + \alpha v_n(bx, b^{-\lambda}t)\}$$
$$= \int_0^t b^{-\lambda} ds' \int_{y\epsilon\Gamma(s)} b^{d-1} d^{d-1}\sigma_{y'}$$
$$\cdot\, G(bx - by, b^{-\lambda}(t - s)) v_n(by', b^{-\lambda}s')$$

Note that the surface integral is over those points y' for which y is in $\Gamma(s)$. Note $y\epsilon\Gamma(s)$ means $y\epsilon\Gamma(b^{-\lambda}s')$ which is identical to $by'\epsilon\Gamma(b^{-\lambda}s')$.With self-similarity

$$y'\epsilon\Gamma(s') \Leftrightarrow by'\epsilon\Gamma(b^{-\lambda}s') \qquad (*)$$

Stage 2. For the Green's for we have the purely algebraic transformation

$$G(b(x-y), b^{-\lambda}t; D) = b^{-d}G(s-y, t; D/b^{2+\lambda})$$

Next, we assume single scale self similarity, so lengths scale as

$$\xi(bx, b^{-\lambda}t) = b\xi(x,t)$$

while all time measurements scale as

$$\Theta(bx, b^{-\lambda}t) = b^{-\lambda}\Theta(x,t)$$

This means that curvature scales as

$$b\kappa(bx, b^{-\lambda}t) = \kappa(x,t)$$

and velocity as

$$v_n(bx, b^{-\lambda}t) = b^{1+\lambda}v_n(x,t)$$

The self similarity assumption is combined with (*) which says that (statistically) if a point y is on the interface at time s , then the point by will be on the interface at time $b^{-\lambda}t$.

In other words, if we keep rescaling time by $b^{-\lambda}$ and space by b then the shape should asymplotically approach a particular shape.

Stage 3. We now use the relations of self similarity above to rewrite the eqn (2):

$$\frac{d_0}{b}\{\kappa(x,t) + \alpha b^{2+\lambda} v_n(x,t)\} \tag{4.3}$$
$$= \int_0^t ds' \int_{y' \epsilon \Gamma(s')} d^{d-1}\sigma_{y'} G(x-y,t;D/b^{2+\lambda}) v_n(y',s')$$

Hence, (4.3) is identical to (4.1) upon rescaling

$$d_0 \to d_0/b; \quad \alpha \to \frac{\alpha}{b^{-2-\lambda}}; \quad D \to \frac{D}{b^{2+\lambda}}. \tag{4.4}$$

Hence, one has (with R as the characteristic length):

OLD	NEW
d_0	d_0/b
D	$D/b^{2+\lambda}$
α	$\alpha/b^{-2-\lambda}$
$\xi(bx, b^{-\lambda}t)$	$b\xi(x,t)$
$R(b^{-\lambda}t, d_0, \alpha, D)$	$bR(t, d_0/b, \alpha/b^{-2-\lambda}, D/b^{2+\lambda})$

By a simple substitution $t_1 = b^{-\lambda}t$, we write

$$R(t_1, d_0, \alpha, D) = bR(b^{\lambda}t_1, d_0/b, \alpha/b^{-2-\lambda}, D/b^{2+\lambda}).$$

Stage 4. Choose $b = t_1^{-1/\lambda}$ (and omit subscript 1) yielding,

$$R(t, d_0, \alpha, D) = t^{-1/\lambda}R(1, d_0/t^{-1/\lambda}, \alpha/t^{2+\lambda}, D/t^{-(2+\lambda)/\lambda}) \tag{4.5}$$

Choosing λ (which was to be determined): Physically trivial if $\lambda > -2$ or $\lambda < -2$, i.e., zero diffusion or infinite diffusion. Nontrivial fixed point is $\lambda = -2$, yielding:

$$R(t, d_0, \alpha, D) = t^{1/2}R(1, d_0/t^{1/2}, \alpha, D). \tag{4.6}$$

CONCLUSIONS.

1. The characteristic length evolves as $R(t) \sim t^{1/2}$ in the fully dynamic (parabolic) case. This is under the assumption that there is no singularity in terms of the capillarity length. In other words, $R(t, 0, \alpha, D)$ is well defined for large t.

2. Capillarity length, $d_{0,}$ is essentially irrelevant for large time – sharp contrast with its stabilizing role in short times. Tne one exception is in the quasi-static case, with d_0 invariant and $R(t) \sim t^{1/3}$ as we see below.

Part 2 – Interface Problems with Quasi-Static Approximation

Recall that we have been considering a material occupying a spatial region, Ω, in d-dimensional space that can be in either of two phases, which we call liquid and solid. The mathematical model consists of determining the temperature, $T(x,t)$, and the interface, $\Gamma(t)$, for

$$\begin{aligned} CT_t &= K\Delta T && \text{in} \quad \Omega\backslash\Gamma \\ lv_n &= -K[\nabla T \cdot \hat{n}]_{-}^{+} && \text{on} \quad \Gamma \\ T &= \frac{-\sigma_0}{[s]_{eq}}(\kappa + \alpha v_n) && \text{on} \quad \Gamma. \end{aligned}$$

Jasnow and Vinals utilized the following conditions: (i) the dynamical undercooling was set to zero ($\alpha = 0$); (ii) one of the two phases was suppressed so that the equations involved one of the phases; (iii) the quasi-static limit was considered by suppressing the time dependence in (i.e., $CT_t = 0$); (iv) a plane wave solution was utilized (through flux conditions) and subtracted from the solutions. Under these conditions they found that the characteristic length, $R(t)$, of a system with single scale self-similarity must have the large time behavior $R(t) \sim t$. Using the same formulation of consolidating the heat equation and the latent heat at the interface, i.e., we have,

$$CT_t - K\Delta T = -\frac{l}{2}\varphi_t.$$

Multiplying the two sides of the equation by $1/K$ and setting $CT_t = 0$, we write, the quasi-static approximation as,

$$\Delta T = \frac{l}{2K}\varphi_t.$$

Treating the phase change as a source term with support along the interface, $\Gamma(t)$, and using the Green's formulation, we write,

$$T(x) = \int_{\Omega} d^d y \; G(\vec{x}-\vec{y})\Big(\frac{l}{2K}\varphi_t(\vec{y},t)\Big) + \int_{\partial\Omega}\Big(T(\vec{y})\frac{\partial G}{\partial \nu}(\vec{x}-\vec{y}) + G(\vec{x}-\vec{y})\frac{\partial T}{\partial \nu}(\vec{y})\Big)d^{d-1}\sigma_y \tag{4.7}$$

$$G(\vec{x}-\vec{y}) = \begin{cases} \frac{1}{2(2-d)\omega_d}|\vec{x}-\vec{y}|^{2-d} & \text{for } d > 2, \\ \frac{1}{2\pi}\log|\vec{x}-\vec{y}| & \text{for } d = 2. \end{cases}$$

Here, the simplest Green's function for infinite domains is implemented. Since we are interested in very large domains, this is a good approximation. In this discussion we consider the case $d \geq 3$. The results for $d = 2$ are similar though more technical.

In a manner similar to the fully-dynamical case we obtain, upon substituting the interface condition,

$$-\frac{\sigma_0}{[s]_{eq}}(\kappa + \alpha v_n(\vec{x},t)) = -\frac{l}{C}\frac{1}{D}\int_{\Gamma\langle t\rangle} d^{d-1}\sigma_y \; G(\vec{x}-\vec{y})v_n(\vec{y},t).$$

In terms of the capillarity length, $d_0 := \frac{\sigma_0/[s]_{eq}}{l/C}$, we write this as,

$$d_0(\kappa + \alpha v_n(\vec{x},t)) = \frac{1}{D}\int_{\Gamma\langle t\rangle} d^{d-1}\sigma_y \; G(\vec{x}-\vec{y})v_n(\vec{y},t). \tag{4.8}$$

We now proceed to rescale and renormalize this expression. Once again,

Stage 1. Algebraic substitution of

$$b\vec{\eta} \quad \text{for} \quad \vec{\eta} \qquad \text{and} \qquad b^{-\lambda}t \quad \text{for} \quad t$$

for arbitrary $b > 0$ and λ in $\mathbf{R}$, yields,

$$d_0\{\kappa(b\vec{\eta}, b^{-\lambda}t) + \alpha v_n(b\vec{\eta}, b^{-\lambda}t)\} = \frac{1}{D}\int_{\Gamma(b^{-\lambda}t)} d^{d-1}\sigma_y \; G(b\vec{\eta}-\vec{y})v_n(\vec{y}, b^{-\lambda}t).$$

Next, we define the new variables,

$$\vec{y}' = y/b \qquad \text{and} \qquad \sigma_{y'} = \sigma_y/b$$

so that the interface equation is transformed into

$$d_0\Big\{\kappa(b\vec{\eta}, b^{-\lambda}t) + \alpha \; v_n(b\vec{\eta}, b^{-\lambda}t)\Big\} = \frac{1}{D}\int_{by'\in\Gamma(b^{-\lambda}t)} b^{d-1}d^{d-1}\sigma_{y'} \; G(b\vec{\eta} - b\vec{y}')v_n(b\vec{y}', b^{-\lambda}t).$$

Note that the surface integral is over those points for which $y \in \Gamma(b^{-\lambda}t)$ which is identical (algebraically) to $by' \in \Gamma(b^{-\lambda}t)$. When we assume self-similarity, the latter will be equivalent to $y' \in \Gamma(t)$.

Stage 2. We examine the scaling of the individual terms. The Green's function scales differently from the fundamental solution:

$$G(b\vec{\eta} - b\vec{y'}) = b^{2-d}G(\vec{\eta} - \vec{y'}).$$

The length scales and time scales are the same as before:

$$\xi(b\vec{\eta}, b^{-\lambda}t) = b\xi(\vec{\eta}, t)$$

$$T(b\vec{\eta}, b^{-\lambda}t = b^{-\lambda}T(\vec{\eta}, t)$$

Also, the length scaling implies $by' \in \Gamma(b^{-\lambda}t) \Leftrightarrow y' \in \Gamma(t)$. Again, curvature is 1/length so it scales as

$$b\kappa(\vec{\eta}, t) = \kappa(b\vec{\eta}, b^{-\lambda}t).$$

Velocity scales as

$$\upsilon_n(b\vec{\eta}, b^{-\lambda}t) = b^{1+\lambda}\upsilon_n(\vec{\eta}, t).$$

With these substitutions, the interface equation now has the form

$$\frac{d_0}{b^{3+\lambda}}\left\{\kappa(\vec{\eta}, t) + \frac{\alpha}{b^{-2-\lambda}}\ \upsilon_n(\vec{\eta}, t)\right\} = \frac{1}{D}\int_{y'\in\Gamma(t)} d^{d-1}\sigma_{y'}\ G(\vec{\eta} - \vec{y'})\upsilon_n\left(\vec{y'}, t\right). \tag{4.9}$$

Stage 3. This interface relation is the same as the original if we rescale the physical parameters as follows:

$$d_0 \to \frac{d_0}{b^{3+\lambda}} \qquad \text{and} \qquad \alpha \to \frac{\alpha}{b^{-2-\lambda}}.$$

In particular, the basic length scale must also satisfy the length scaling relationship,

$$R(b^{-\lambda}t; \alpha, d_0) = bR\left(t; \alpha/b^{-2-\lambda}, d_0/b^{3+\lambda}\right).$$

Stage 4. At this point, we substitute, $t = b^{\lambda}t_1$, since b is arbitrary, and write this as,

$$R(b^{-\lambda}t; \alpha, d_0) = bR\left(t; \alpha/b^{-2-\lambda}, d_0/b^{3+\lambda}\right),$$

or, using $t_1 = b^{-\lambda}t$, we write this as,

$$R(t;\alpha,d_0) = t^{-1/\lambda}R\left(1;\alpha/t^{(2+\lambda)/\lambda},d_0/t^{-(3+\lambda)/\lambda}\right).$$

Analysis of the parameter λ. The value of λ clearly determines the long time asymptotics of the characteristic length, $R(t)$, provided that R is not singular in the second and third variables at 0. Both the cases $\lambda < -3$ and $\lambda > 0$ lead to the result that d_0 approaches ∞ as $t \to \infty$. These, however, yield fixed points that are physically not meaningful. Similarly, if $\lambda \in (-2,0)$, then α approaches ∞ for large t, while d_0 approaches 0 which yield also the nonphysical fixed points. Hence, any possible value for λ which yields the nontrivial fixed point lies in the interval $[-3,-2]$. This indicates that the characteristic length, $R(t)$, increases as $t^{-1/\lambda}$ as λ varies in the continuous spectrum $[-3,-2]$.

The result also confirms, once again, for $\lambda \in (-3,-2]$ that the capillarity length, d_0, is essentially irrelevant for large time, which is sharp contrast with its stabilizing role for short times (see references. The only exception is the value $\lambda = -3$ (i.e. $R(t) \sim t^{1/3}$) for which the capillarity length, d_0, is invariant for large time. In this case, the scaling does not depend on the non-singularity of R as a function of d_0.

FURTHER RESEARCH.

1. Renormalization in a dynamical setting –Green's function and related DE methods may be the key. Transition: equil → dynamics. (C&Merdan)

2. Transition between 2D and 3D: Green's function and PDEs.

3. Understanding a broad range of interface phenomena through this approach.

4. Understanding the transition between short time (linear stability) and the long term asymptotics (through RG). Crossover behavior.

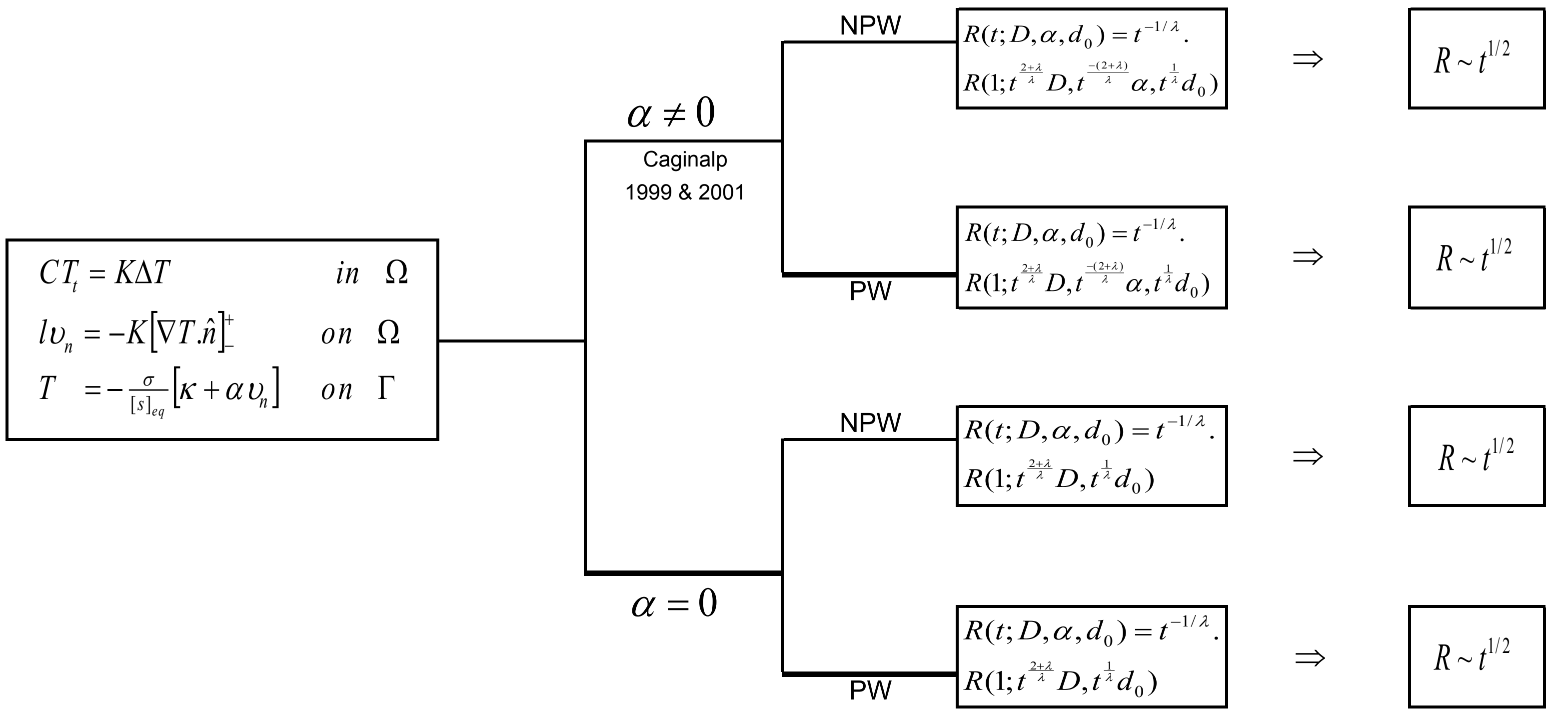

Figure 2. **NPW** : There is not a plane wave imposed, **PW** : There is a plane wave imposed.

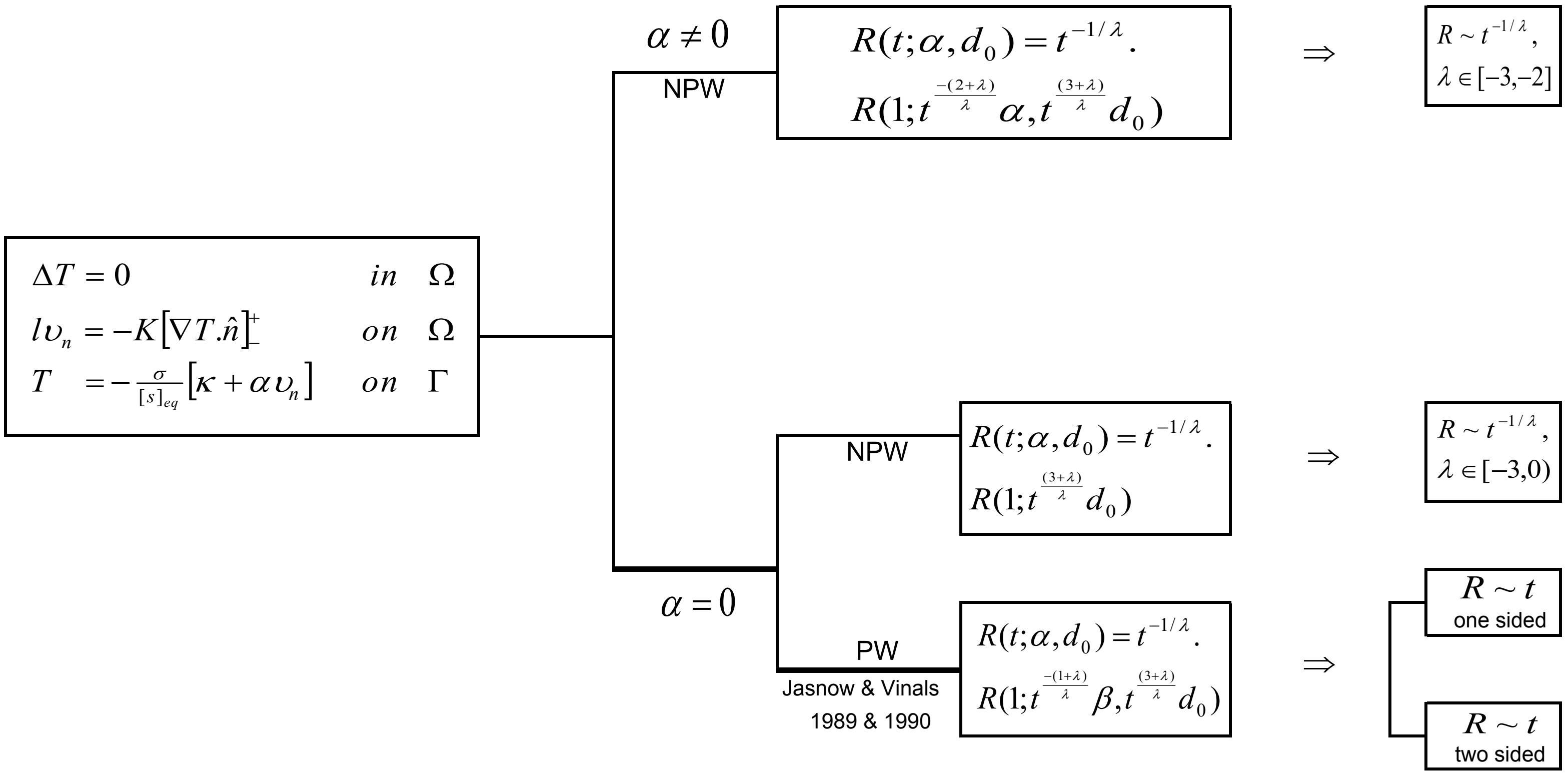

Figure 1. **NPW** : There is not a plane wave imposed, **PW** : There is a plane wave imposed.

RANDOM WALK

Exercise 1. Consider the random walk with a Gaussian probability distribution $p(r_i)$ for each of n steps. Write the probability distribution for n steps, $\overline{P}(r')$, and carry out the integral. Let

r_i := *ith step vector*

$$\overline{P}(r') = \int_{\mathbb{R}^d} .. \int_{\mathbb{R}^d} d^d r_1 \ldots d^d r_n \delta\left(r' - \sum_{i=1}^{n} r_j \right) p(r_1) \ldots p(r_n) \tag{A}$$

$$p(r_i) = \left(2\pi\sigma_o^2\right)^{-\frac{d}{2}} \exp\left(\frac{-|r_i|^2}{2\sigma_o^2} \right) \tag{B}$$

We want to show that the integral (A) is the same form as (B) with σ_o^2 replaced by $n\sigma_o^2$.

Solution. Write the delta function in the form (Fourier transform of constant function)

$$\delta(r) = (2\pi)^{-d} \int_{\mathbb{R}^d} d^d k e^{ik \cdot r} \tag{1}$$

so that (A) yields

$$\overline{P}(r') = \int_{\mathbb{R}^d} .. \int_{\mathbb{R}^d} d^d r_1 \ldots d^d r_n (2\pi)^{-d} \int_{\mathbb{R}^d} d^d k e^{ik \cdot (r' - \sum_{i=1}^{n} r_j)} p(r_1) \ldots p(r_n) \tag{2}$$

(Note that we do this in a formal manner. In order to make it rigorous, one needs to approximate the delta function with smooth functions or use the theory of distributions.)The factor $e^{ik \cdot r'}$ is indep of r and can be moved outside the $dr_1 \ldots dr_n$ integrals (but not the dk integral), so

$$\overline{P}(r') = \int_{\mathbb{R}^d} d^d k e^{ik \cdot r'} \int .. \int d^d r_1 \ldots d^d r_n (2\pi)^{-d} e^{-ik \cdot \sum_{j=1}^{n} r_j} p(r_1) \ldots p(r_n) \tag{3}$$

The r_i intergals decouple so that (3) can be written as

$$\overline{P}(r') = \int_{\mathbb{R}^d} d^d k \frac{e^{ik \cdot r'}}{(2\pi)^d} \prod_{j=1}^{n} \int_{\mathbb{R}^d} d^d r_j e^{-ik \cdot r_j} p(r_j) \tag{4}$$

We need to evaluate the dr_j integrals in (4) as:

$$I_j(k) := \int_{\mathbb{R}^d} d^d r_j e^{-ik\cdot r_j} \left(2\pi\sigma_o^2\right)^{\frac{-d}{2}} \exp\left(\frac{-|r_j|^2}{2\sigma_o^2} \right)$$

$$= \prod_{m=1}^{d} \int_{-\infty}^{\infty} dr_j^{(m)} e^{-ik^{(m)} r_j^{(m)}} \exp\left(\frac{-(r_j^{(m)})^2}{2\sigma_o^2} \right) \left(2\pi\sigma_o^2\right)^{\frac{-d}{2}} \tag{5}$$

$$\text{with } \vec{k} := (k^{(1)}, \ldots k^{(m)}, \ldots k^{(d)}),\ r_j = (r_j^{(1)}, \ldots r_j^{(d)})$$

We know the definite integral

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)} dx = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{(4a)}} \qquad a = \frac{1}{2\sigma_o^2} \qquad b = ik^{(m)}$$

$$(2\pi\sigma_o^2)^{\frac{d}{2}} I_j(k) = \prod_{m=1}^{d} (2\pi\sigma_o^2)^{\frac{1}{2}} e^{-k^{(m)^2} \frac{\sigma_o^2}{2}} \tag{6}$$

The factors of $(2\pi\sigma_o^2)^{\frac{d}{2}}$ cancel in (6) so that the entire integral $I_j(k)$ is given by

$$I_j(k) = e^{-|k|^2 \frac{\sigma_o^2}{2}}$$

so (4) becomes

$$\overline{P}(r') = \int_{\mathbb{R}^d} d^d k \frac{e^{ik\cdot r'}}{(2\pi)^d} e^{-|k|^2 \sigma_o^2 \frac{n}{2}} \tag{7}$$

$$\text{(factor of } n \text{ from product } \prod_{m=1}^{n} \text{)}$$

This is the same as Creswick's (1.6.4) except for the sign of $ik \cdot r'$ due to the convention in the definition of the function.

Also, they change the variable from r' *to* r in (1.6.4).
Finally, we can integrate (7) using the formula above:

$$\text{let } a := \sigma_o^2 \frac{n}{2},\quad b := ir'^{(m)}$$

$$\overline{P}(r') = \frac{1}{(2\pi)^d} \prod_{m=1}^{d} \left(\frac{2\pi}{n\sigma_o^2} \right)^{\frac{1}{2}} \exp\left(\frac{-r'^{(m)2}}{2\sigma_o^2 n} \right)$$

$$= \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{1}{n\sigma_o^2} \right)^{\frac{d}{2}} \exp\left(\frac{-|r'|^2}{2n\sigma_o^2} \right)$$

Note factor of 2 typo in Creswick. Therefore

$$\overline{P}(r') = \frac{1}{(2\pi n\sigma_o^2)^{\frac{d}{2}}} \exp\left(-\frac{|r'|^2}{2n\sigma_o^2} \right)$$

Then coarse graining (i.e., r' replacing $\sum_{i=1}^{n} r_j$) and renormalizing σ by

$$\sigma' = n^{\frac{1}{2}}\sigma_o$$

yields the same prob dist.

PERCOLATION EXERCISES.

Exercise 1. Majority rules on a single triangle.

We consider a triangular lattice, and do coarse graining by taking majority rules. If all three are occupied, i.e., probability p^3, then it is occupied in the new lattice. If any two are occupied, i.e., probability $3p^2(1-p)$, then it is also occupied. This means that the transforemation is given by

$$\overline{p} = F(p) = p^3 + 3p^2(1-p) = -2p^3 + 3p^2.$$

The critical probability, p_c, is given by the point at which

$$p_c = -2p_c^3 + 3p_c^2$$

The solution is : $\{p_c = 0\}, \{p_c = .5\}, \{p_c = 1.0\}$. Clearly, only $p_c = .5$ is non-trivial, and is our desired critical probability.

Note that the scale is twice the distance from any vertex to the opposite side, denoted $b = 2a$. This is because by averaging the triangles, we go from the perpendicular bisectors of one triangle through another to the perpendicular bisectors of the third. The distance a is given by the right triangle that is half of the original equilateral triangle as

$$a^2 = 1 - \left(\frac{1}{2}\right)^2 \text{ or } \quad a = \frac{\sqrt{3}}{2}.$$

so we have $b = 2a = (\frac{1}{2} + 1 + \frac{1}{2})\sqrt{3}/2 = \sqrt{3}$. In other words, the new centers of the triangles are a distance $b = \sqrt{3}$ away from each other, where they were previously just 1, so this is the new scale.

Using this scale, we can determine the critical exponent. Implementing the formula derived earlier, one has:

$$\nu = \frac{\ln b}{\ln F'(p_c)};$$

$$F'(p) := -6p^2 + 6p \text{ so } \quad F'(1/2) = \frac{3}{2};$$

$$\nu = \frac{\ln \sqrt{3}}{\ln(3/2)} = 1.3548.$$

Exercise 2. (Optional)

If we take three triangles side by side, then a connection is possible going left to right if (a) all five are occupied, i.e., probability p^5; (b) any four are occupied, i.e., p^4q ; (c) any three are occupied so long as it is not the left two or right two that are unoccupied, so we have $\binom{5}{3} - 2 = 8$ ways of having p^3q^2; (d) the top two are occupied and the bottom three unoccupied, i.e., p^2q^3. Then $F(p)$ is given by:

$$F(p) = p^5 + 5p^4(1-p) + 8p^3(1-p)^2 + p^2(1-p)^3$$

Once again the critical probability satisfies the relation,

$$p_c = F(p_c)$$

namely,

$$p^5 + 5p^4(1-p) + 8p^3(1-p)^2 + p^2(1-p)^3 - p = 0$$

whose solution is : $\{p = 0\}, \{p = -.40147\}, \{p = .54518\}$,
$\{p = 1.0\}, \{p = 1.5229\}$. The only solution $p_c \in (0,1)$ is

$$p_c = .54518.$$

The scale is not as clearly defined in this perspective, so it is more difficult to calculate the exponent.

Decay Exercises.

Exercise 1. Verification of (2.6) to (2.8):

$$u(x,t) = \int_{-\infty}^{\infty} dy G(x-y) g(y) + \varepsilon(\dots)$$

To leading order in ε, we substitute the initial condition, $g(y)$, and integrate (keeping in mind that l is small):

$$u(x,t) = \int_{-\infty}^{\infty} dy \frac{\exp(-(x-y)^2/2t)}{(2\pi t)^{1/2}} Q_0 \frac{e^{-y^2/(2l^2)}}{(2\pi l^2)^{1/2}}$$

$$u(x,t) = Q_0 (2\pi t^{1/2} l)^{-1} \exp(-x^2/(2t)) \int_{-\infty}^{\infty} dy \exp(2xy/(2t)) \exp\left(-y^2\left(\frac{1}{2t} + \frac{1}{2l^2}\right)\right)$$

Note that for small l the factor $e^{-y^2/(2l^2)}$ will be very close to zero except when y is near zero. Hence, the term

$$\exp(2xy/(2t)) \cong 1 + \frac{2xy}{2t} + O(y^2)$$

can be neglected except for the constant 1. Using the identity

$$\int_{-\infty}^{\infty} dz \exp(-az^2) = \left(\frac{\pi}{a}\right)^{1/2}$$

we can write, with $a := (t + l^2)/2tl^2$,

$$u_0(x,t) \cong Q_0 (2\pi t^{1/2} l)^{-1} \exp(-x^2/(2t)) \frac{2^{1/2} \pi^{1/2} t^{1/2} l}{(t + l^2)^{1/2}}$$

Also, we can write

$$\exp(-x^2/(2t)) \cong \exp\left(\frac{-x^2}{2(t + l^2)}\right)$$

since l^2 is small compared with t. Then we have,

$$u_0(x,t) \cong \frac{Q_0}{[2\pi(t + l^2)]^{1/2}} \exp\left(\frac{-x^2}{2(t + l^2)}\right).$$

Note that this expression (i.e., (2.8) of paper) is just the leading order in ε, and within that order in ε, it is also the leading order in l (i.e., neglecting the orders l^2 and higher)

Exercise 2. Calculating the u_1 term. Recall (2.6):

$$u(x,t) = \int_{-\infty}^{\infty} dyG(x-y,t)g(y) + \varepsilon \int_0^t ds \int_{-\infty}^{\infty} dyG(x-y,t-s)N[y,u(y,s)\ldots]$$

The first integral involving the initial condition generated the u_0 term. Since the second is multiplied by $\varepsilon,$ we need only substitute u_0 into the expression for N in place of u in order to generate the u_1 term. One has:

$$\varepsilon u_1(x,t) = \varepsilon \int_0^t ds \int_{-\infty}^{\infty} dyG(x-y,t-s)N[y,u_0(y,s)\ldots]$$

$$u_1(x,t) = \int_0^t ds \int_{-\infty}^{\infty} dy \frac{\exp(-(x-y)^2/2(t-s))}{(2\pi(t-s))^{1/2}} N[y,u_0(y,s)\ldots]$$

Now using

$$N[y,u_0(y,s)\ldots] = y^m u_0^n (\partial u_0/\partial x)^p = y^m u_0^{1-p} (\partial u_0/\partial x)^p = y^m u_0^{1-p} \left(\frac{-y}{s+l^2}\right)^p u_0^{1-p}$$

due to the dimensional relation $n = 1-p$ and the identity:

$$\frac{\partial u_0}{\partial x} = \left(\frac{-x}{t+l^2}\right) u_0$$

we obtain:

$$u_1(x,t) = \int_0^t ds \int_{-\infty}^{\infty} dy \frac{\exp(-(x-y)^2/2(t-s))}{(2\pi(t-s))^{1/2}} y^m \frac{Q_0}{[2\pi(s+l^2)]^{1/2}} \exp\left(\frac{-y^2}{2(s+l^2)}\right)\left(\frac{-y}{s+l^2}\right)^p.$$

Next, we want to approximate this integral by extracting the leading order in l. Note that the term

$$\exp\left(\frac{-y^2}{2(s+l^2)}\right)$$

will be very small when y is not near zero. Hence we can replace $(x-y)$ by x (recall we are considering large x and t). Also, we are interested in obtaining the leading order in l, which will arise from the $s \approx 0$ end of the integral. This line of reasoning is known as Laplace's method (see Erdelyi text).

With these substitutions one has:

$$u_1(x,t) \cong \frac{\exp(-x^2/2t)}{(2\pi t)^{1/2}} \int_0^t ds \int_{-\infty}^{\infty} dy$$
$$y^m \frac{Q_0}{[2\pi(s+l^2)]^{1/2}} \exp\left(\frac{-y^2}{2(s+l^2)}\right)\left(\frac{-y}{s+l^2}\right)^p.$$

so that rewriting this, one has:

$$u_1(x,t) = \frac{(-1)^p Q_0}{2\pi} \frac{\exp(-x^2/2t)}{t^{1/2}} \int_0^t ds(s+l^2)^{-p-1/2}$$
$$\int_{-\infty}^{\infty} dy \exp\left(\frac{-y^2}{2(s+l^2)}\right) y^{m+p}$$
$$+ \text{ terms that are smaller in terms of } l^{-1}.$$

The integrand of the s integral can be written in terms of

$$w := y/(s+l^2)^{1/2}$$

as

$$I_1 \ := (s+l^2)^{-1} \int_{-\infty}^{\infty} w^{2(p-1)} e^{-w^2/2} dw$$
$$= \frac{(2\pi)^{1/2}}{s+l^2} \{1 \cdot 3 \cdot \cdot \cdot |2(p-1)-1|\}$$

Hence, the only remaining integral is essentially $\int_0^t ds(s+l^2)^{-1}$. Combining the $O(\varepsilon)$ term with the u_0 found earlier we can write the solution to $O(\varepsilon)$ and to leading order in l as:

$$u(x,t;\varepsilon,l) = \frac{Q_0 t^{-1/2}}{(2\pi)^{1/2}} \exp(-x^2/2t)$$
$$\left\{1 + \varepsilon(-1)^p 1 \cdot 3 \cdot \cdot \cdot |2p-3| \log(t/l^2)\right\}$$

for $p \geq 1$.

Exercise 3. Second order nonlinearities. Determine u_1 for the nonlinearity

$$x^m u^n u_x^p u_{xx}^q$$

where the following dimensional identities apply:

$$n + p + q = 1 \quad \text{and} \quad p + 2q - m = 2.$$

Solution. As in the nonlinearities for u and u_x we substitute the leading order solution

$$u_0(x,t) \simeq \frac{Q_0}{[2\pi(t+l^2)]^{1/2}} \exp\left(\frac{-x^2}{2(t+l^2)}\right)$$

into the integral expression, (2.6). Note that

$$\frac{\partial^2}{\partial x^2} u_0(x,t) = -(t+l^2)^{-1}\left(\frac{x^2}{t+l^2} - 1\right) u_0(x,t)$$

The u_1 term will arise from the second integral of (2.6) when u_0 is substituted into $N[\dots]$. Note that the nonlinearity will be (as a function of y and s):

$$\begin{aligned} y^m u^n u_y^p u_{yy}^q &\simeq y^m u_0^n u_0^p \left(\frac{-y}{s+l^2}\right)^p u_0^q \left\{(s+l^2)^{-1}\left(\frac{y^2}{s+l^2} - 1\right)\right\}^q \\ &= (-1)^p u_0 \left(\frac{y}{(s+l^2)^{1/2}}\right)^p (s+l^2)^{-q} \left(\frac{y^2}{s+l^2} - 1\right)^q \end{aligned}$$

Letting $w := y/(s+l^2)^{1/2}$ we can rewrite this as

$$y^m u^n u_y^p u_{yy}^q \simeq (-1)^p \left\{w(s+l^2)^{1/2}\right\}^m u_0 w^p (s+l^2)^{-q} \left(w^2 - 1\right)^q.$$

Note that the exponent of $(s+l^2)$ is $m/2 - q = p/2 - 1$ by the dimensional identities. Then we have, similar to the first order nonlinearity

$$\begin{aligned} u_1(x,t) = \frac{(-1)^p Q_0}{2\pi} \frac{\exp(-x^2/2t)}{t^{1/2}} \int_0^t ds (s+l^2)^{-1} \\ \int_{-\infty}^{\infty} w^{p+m}(w^2-1)^q e^{-w^2/2} dw \end{aligned}$$

Expanding the $(\dots)^q$ term, we can evaluate the integral and obtain

$$u(x,t) = Q_0 \frac{\exp(-x^2/2t)}{(2\pi t)^{1/2}}$$
$$\left\{ 1 + \varepsilon \sum_{j=0}^{q} (-1)^{j+p} 1 \cdot 3 \cdots |2p + 4q - 2j - 3| \log(t/l^2) \right\}$$

Exercise 4. Verification of RG in a special case. Consider

$$u_t = \frac{1}{2} u_{xx} + \varepsilon x^{-1} u_x$$

and show that the RG decay exponent of $-\varepsilon$ (plus the classical 1/2) is attained, i.e., the solution behaves as $t^{-1/2-\varepsilon}$.

Solution. For any equation of the form

$$u_t = \frac{1}{2} u_{xx} + \varepsilon F[x, u, u_x, u_{xx}]$$

we can use the substitutions

$$u(x,t) =: e^{\phi(\xi,\tau)}, \quad \tau := \log(t + t_0), \quad \xi := x(t + t_0)^{-1/2}.$$

Then the DE can be written as

$$\phi_\tau = \frac{1}{2} \left[\phi_{\xi\xi} + \phi_\xi^2 + \xi \phi_\xi \right] + \varepsilon F[\xi, 1, \phi_\xi, \phi_{\xi\xi} + \phi_\xi^2].$$

(Note that the $e^\phi (t + t_0)^{-1/2}$ terms cancel as we transform into the τ and ξ variables.)

For the particular source term $\varepsilon x^{-1} u_x$ we have the transformed equation,

$$\phi_\tau = \frac{1}{2} \left[\phi_{\xi\xi} + \phi_\xi^2 + \xi \phi_\xi \right] + \varepsilon \xi^{-1} \phi_\xi.$$

We try a solution of the form (for some α, σ in R)

$$\phi(\xi, \tau) = \sigma \xi^2 + \alpha \tau$$

which corresponds to $u(x,t) = \exp[\phi(\xi,\tau)] = t^\alpha \exp\left(\frac{\sigma x^2}{t}\right)$. Then substitution of this form of $\phi(\xi,\tau)$ yields the following algebraic identity:

$$\alpha = (1 + 2\varepsilon)\sigma + (1 + 2\sigma)\sigma\xi^2.$$

The coefficient of ξ^2 must vanish and we have $\sigma = -1/2$. Then solving the constant term for α we have $\alpha = -1/2 - \varepsilon$. Thus the non-negative exact solution to our differential equation must be

$$u(x,t) = t^{-1/2-\varepsilon} \exp\left(\frac{x^2}{2t}\right).$$

Hence, this confirms the RG exponent.

Exercise 5. Verify the RG exponent with the special case

$$u_t = \frac{1}{2}u_{xx} + \varepsilon u^{-1}u_x^2$$

using the same (ξ, τ) transformation.

Solution. The same transformation as above yields the transformed equation

$$\varphi_\tau = \frac{1}{2}\left[\varphi_{\xi\xi} + \varphi_\xi^2 + \xi\varphi_\xi\right] + \varepsilon\varphi_\xi^2.$$

The substitutions above yield the non-negative exact solution to our DE,

$$u(x,t-t_0) = t^{\frac{-1}{2(1-2\varepsilon)}} \exp\left(\frac{-x^2}{(1-2\varepsilon)2t}\right). \qquad (*)$$

Note: Letting $u(x,t) = [w(x,t)]^{\frac{1}{(1-2\varepsilon)}}$, where $\varepsilon \neq \frac{1}{2}$, one can transform (4.9) into linear (diffusion equation) form, $i.e.$ $w_t = \frac{1}{2}w_{xx}$. Hence, the exact solution $(*)$ could be obtained by using the fundamental solution to linear equation, namely $\Gamma(x,t) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)$.

This exact solution agrees with the RG calculations.