MAGIC059: Dynamical Systems I (flows)

1. Outline and Introduction

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http://www.maths-magic.ac.uk/course.php?id=256

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Outline Syllabus (10 hours)

- Definition of a flow (ODE), invariant sets, limit sets
- The Poincaré map
- Linearisation, equilibria, periodic orbits, stability
- Structural stability, Hartman-Grobman Theorem, stable and unstable manifolds
- Centre manifold theorem, local bifurcations of equilibria and periodic orbits, Birkhoff normal form transformations
- The saddle-node-Hopf bifurcation (or other example)
- Global bifurcations in two dimensions: derivation of the Poincaré map, leading on to Dynamical Systems II (maps)
- If there is time: brief discussion of dynamics of dissipative PDEs and the role of symmetry
- If there is time: numerical and symbolic methods for ODEs and a mention of packages available. Continuation and the implicit function theorem

Course Structure and Assessment

- Lectures: Tuesdays 9–10.
- Dates: 8 October – 14 December 2010 (with lecture on 6th November rescheduled)
- Lecturer: Alastair Rucklidge `A.M.Rucklidge at leeds.ac.uk`
- Three examples sheets. Solutions will be provided individually on request.
- I may schedule a tutorial at the end of term
- Assessment: end of semester take home exam
- Course web page: `http://www.maths-magic.ac.uk/course.php?id=256`
  - Handout version of the lecture notes
  - Examples sheets
  - Announcements, discussion forum etc.

Recommended Books I

Main text:

- Kuznetsov *Elements of Applied Bifurcation Theory* Springer 2004

Other texts:

- Guckenheimer and Holmes *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* Springer 1983
- Glendinning *Stability, Instability and Chaos* Cambridge University Press 1994
- Hirsch, Smale and Devaney *Differential Equations, Dynamical Systems, and an Introduction to Chaos* Academic Press 2003
- Strogatz *Nonlinear Dynamics and Chaos* Westview Press 1994
- Lynch *Dynamical Systems with Applications using Maple/MATLAB/Mathematica* Birkhäuser 2004–09
- Wiggins *Global Bifurcations and Chaos* Springer 1988

Recommended Books II

- Wiggins *Introduction to Applied Nonlinear Dynamical Systems and Chaos* Springer 1990
- Parker & Chua *Practical Numerical Algorithms for Chaotic Systems* Springer 1989
- Manneville *Dissipative Structures and Weak Turbulence* Academic Press 1990
- Hoyle *Pattern Formation: An Introduction to Methods* Cambridge University Press 2006
- Sparrow *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors* Springer 1982
- Arrowsmith and Place *Introduction to Dynamical Systems* Cambridge University Press 1990
- Katok and Hasselblatt *Introduction to the Modern Theory of Dynamical Systems* Cambridge University Press 1995
- Golubitsky, Stewart and Schaeffer *Singularities and Groups in Bifurcation Theory. Volume II* Springer 1988

Online sources I

- Herod (Rutgers): Java applet for 2D phase plane `http://www.math.rutgers.edu/courses/ODE/sherod/phase-local.html`
- Mansfield/Beukers (Penn State): Java applet for 2D phase plane `http://www.math.psu.edu/melvin/phase/newphase.html`
- Polking (Rice): Java applet (and matlab) for 2D phase plane `http://math.rice.edu/~dfield/dfpp.html`
- Michael Cross (Caltech): applet for the Lorenz equations `http://crossgroup.caltech.edu/chaos_new/Lorenz.html`
- SIAM DSweb: Dynamical systems software `http://www.dynamicalsystems.org/sw/sw/`
- SIAM DSweb: Tutorials `http://www.dynamicalsystems.org/tu/tu/`

Introduction I

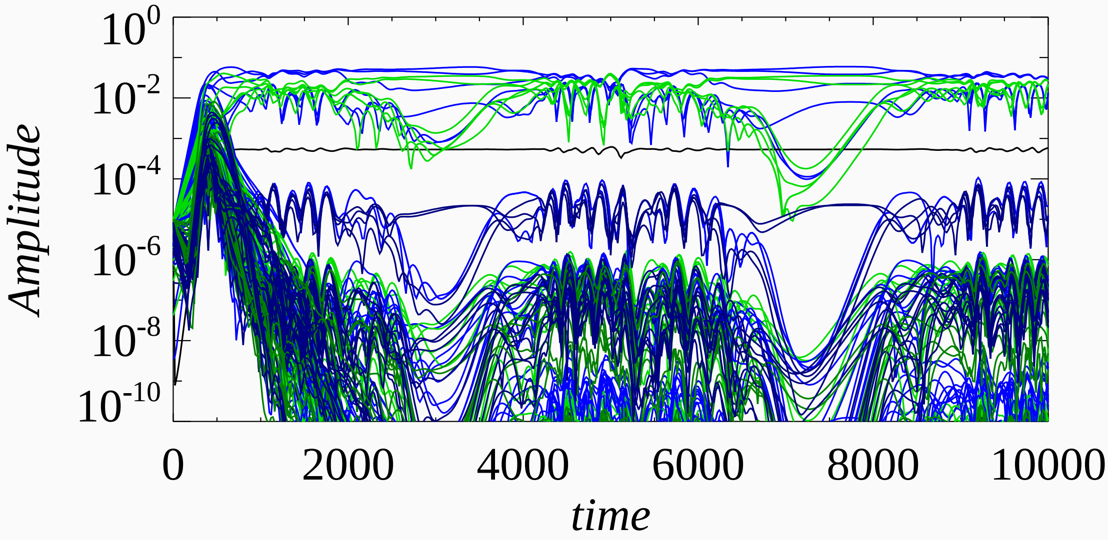
- The aim of this course is to provide a foundation for understanding dynamical systems: the study of systems whose state evolves in time according to some rule.
- Many problems in Applied Mathematics are nonlinear and described by nonlinear ordinary (or partial) differential equations. We will investigate the tools and techniques needed to understand the dynamics that might be found in such systems, and how the behaviour of these systems depends on their parameters.
- The emphasis will be on concepts and examples rather than theorems and proofs, and will include a brief survey of useful numerical methods and packages.
- Students are invited to submit examples of their own for possible discussion.

Introduction II

We will concentrate on:

- flows (ordinary differential equations). Dynamical Systems II (MAGIC060) will concentrate on maps.
- dissipative dynamical systems. Integrable Systems (MAGIC067) treats integrable dynamical systems.
- deterministic dynamical systems. MAGIC065 treats stochastic dynamical systems.
- finite-dimensional dynamical systems. There are some examples of infinite-dimensional dynamical systems (PDEs) in MAGIC014.

Examples of dynamical systems I



Time series: nonlinear surface wave interactions.

Rucklidge and Skeldon, preprint (2010).

Examples of dynamical systems II

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Phase portrait: brain computation with transients. A model of how neural networks in the locust antennal lobe process information.

Rabinovich, Huerta and Laurent, *Transient Dynamics for Neural Processing*, Science **321** 48–50 (2008).

Copyrighted figure omitted

Bifurcation diagram: stable and unstable steady states for an Endex-coupled carboniser–calciner housing the $CaO/CaCO_3$ surface reactions, in the well-stirred fully insulated approximation.

Ball and Sceats, *Separation of carbon dioxide from flue emissions using Endex principles* Fuel **89** 2750–2759 (2010).

Definitions I

Dynamical System: in many systems, the future state of the system can be predicted knowing its current state and the law that governs the system's evolution though time.

The state space (or phase space) X is the set of all possible states $x \in X$ of the system. Here, we can think of $X = \mathbb{R}^n$.

Time t is either continuous ($t \in \mathbb{R}_+$) or discrete ($t \in \mathbb{Z}_+$). If the system is invertible, then $t \in \mathbb{R}$ or $t \in \mathbb{Z}$. The set of allowed values of t will be called T.

The evolution law $\Phi^t(x_0)$ of the system is a mapping from $X \to X$: given that at time $t = 0$ the system is in state x_0, then at time $t \in T$, the state $x(t)$ at time t is determined by

$$x(t) = \Phi^t(x_0).$$

Clearly $\Phi^0(x_0) = x_0$ for all $x_0 \in X$.

Definitions II

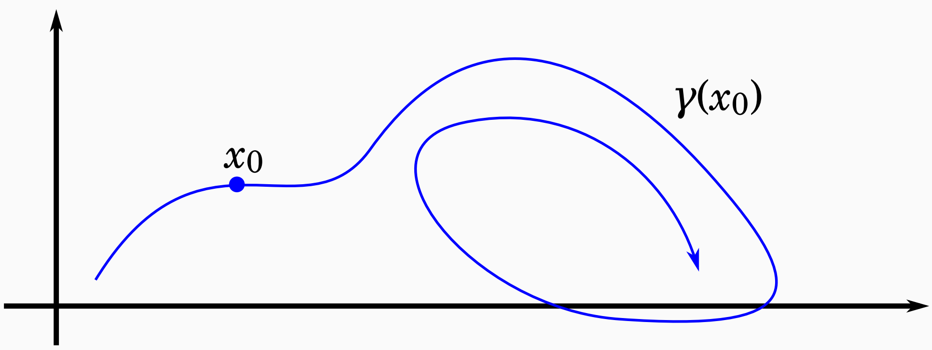
If the system is autonomous, that is, the evolution law does not depend on our choice of the origin of the time coordinate, we get:

$$x(t_1 + t_2) = \Phi^{t_1+t_2}(x_0) = \Phi^{t_1}\left(\Phi^{t_2}(x_0)\right)$$

In the case of continuous time, the set of all $\Phi^t(x_0)$, for all $t \in T$ and $x_0 \in X$, is called a flow (if $T = \mathbb{R}$), or a semi-flow ($T = \mathbb{R}_+$).

The orbit or trajectory of a point x_0 is

$$\gamma(x_0) = \{\Phi^t(x_0) \quad \forall t \in T\}$$



Definitions III

Given $x(t) = \Phi^t(x_0)$, we can show that $x(t)$ obeys an ordinary differential equation (ODE):

$$\frac{dx}{dt} = \dot{x} = \frac{d}{dt}\Phi^t(x_0) = \lim_{\delta t \to 0} \frac{\Phi^{\delta t}(x(t)) - x(t)}{\delta t} = f(x(t)).$$

In the autonomous case, $f(x)$ does not depend explicitly on time.

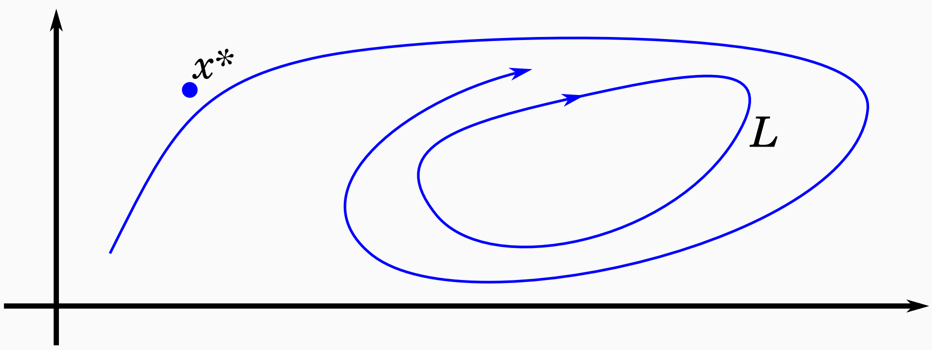
Conversely, an ODE defines a flow, provided solutions exist for all time. Indeed, this is the usual way to define a flow.

Note: throughout this course, we will assume that spaces (such as X) have any extra feature that might be required (for example, a metric), and that everything is smooth enough that any quantity can be differentiated as much as we like. In this case, the existence, uniqueness and smooth dependence theorems for ODEs imply that the flow is invertible.

Definitions IV

A point $x^* \in X$ is called an equilibrium point (or fixed point) if $\Phi^t(x^*) = x^*$ for all $t \in T$. Equilibria satisfy $f(x^*) = 0$.

A closed loop $L \subset X$ is called a periodic orbit if L is not an equilibrium point and every point $x \in L$ satisfies $x = \Phi^{T^*}(x)$ for some $T^* \in T$, $T^* > 0$. The smallest $T^* > 0$ for which this is true is called the period of the periodic orbit.



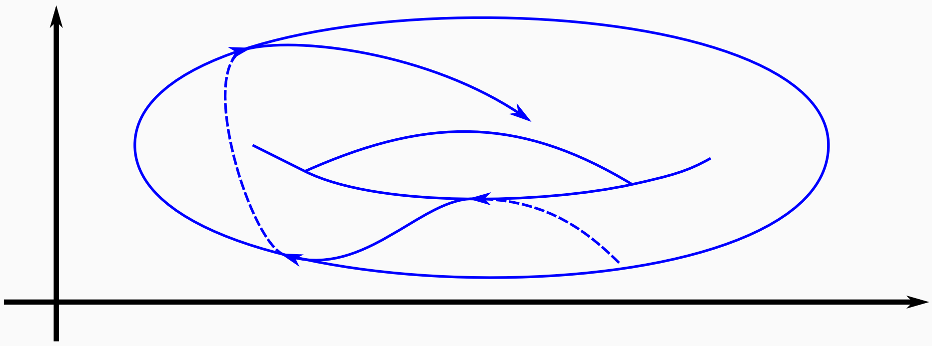
The phase portrait of a dynamical system is a partitioning of its phase space into equilibria, periodic orbits, and other invariant sets.

Definitions V

An invariant set $I \subset X$ is a subset of X such that $x \in I$ implies $\Phi^t(x) \in I$ for all $t \in T$. This implies that $\Phi^t(I) \subseteq I$ for all $t \in T$, and that $\Phi^t(I) = I$ in the invertible case.

Examples of invariant sets:

- Equilibrium points, periodic orbits, orbits.
- Invariant manifolds, tori, α- and ω-limit sets...



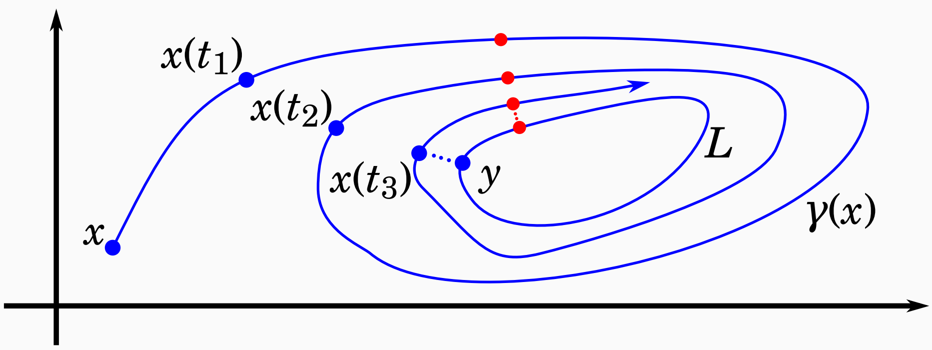
Definitions VI

A point $y \in X$ is called an ω-limit point of $x \in X$ if there is a sequence $t_i \in T$, $i \in \mathbb{Z}_+$, such that $t_i \to \infty$ as $i \to \infty$ and

$$\lim_{i \to \infty} \Phi^{t_i}(x) = y.$$

The set of all ω-limit points of x is called the ω-limit set of x, and is invariant.

Similarly, α-limit points and α-limit sets are defined with $t_i \to -\infty$.



Definitions VII

An invariant set I is called Lyapunov stable if for any sufficiently small neighbourhood $U \supset I$, there is a neighbourhood $V \supset I$ such that $\Phi^t(x) \in U$ for all $x \in V$ and for all $t > 0$. In other words, if you start close to the invariant set ($x \in V$), you stay close ($\Phi^t(x) \in U$).

An invariant set I is called asymptotically stable if there exists a neighbourhood $V \supset I$, such that $\Phi^t(x) \to I$ as $t \to \infty$ for all $x \in V$, as $t \to \infty$. In other words, if you start close to the invariant set ($x \in V$), you tend towards the invariant set ($\Phi^t(x) \to I$).

An invariant set I is called stable if it is Lyapunov stable and asymptotically stable.

Stability is only possible if the flow is dissipative, i.e., if

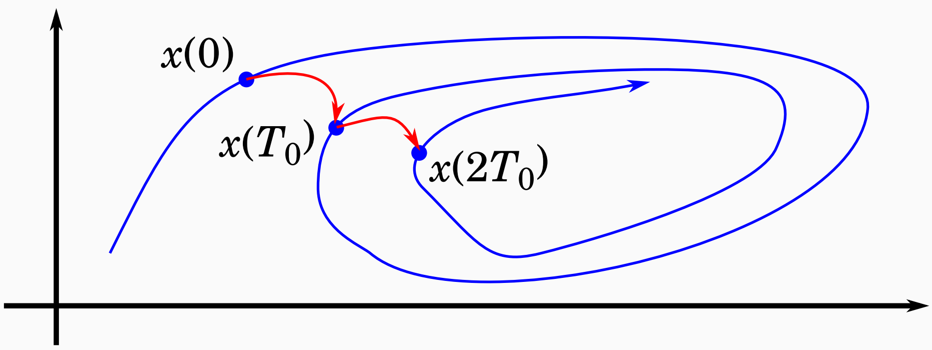
$$\nabla \cdot f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} < 0 \qquad \text{(volumes in phase space contract).}$$

Poincaré map I

The Poincaré map is a method of converting a flow (continuous time) to a map (discrete time). In the simplest case, pick a time $T_0 > 0$ and define a map $S_{T_0} : \mathbb{R}^n \to \mathbb{R}^n$:

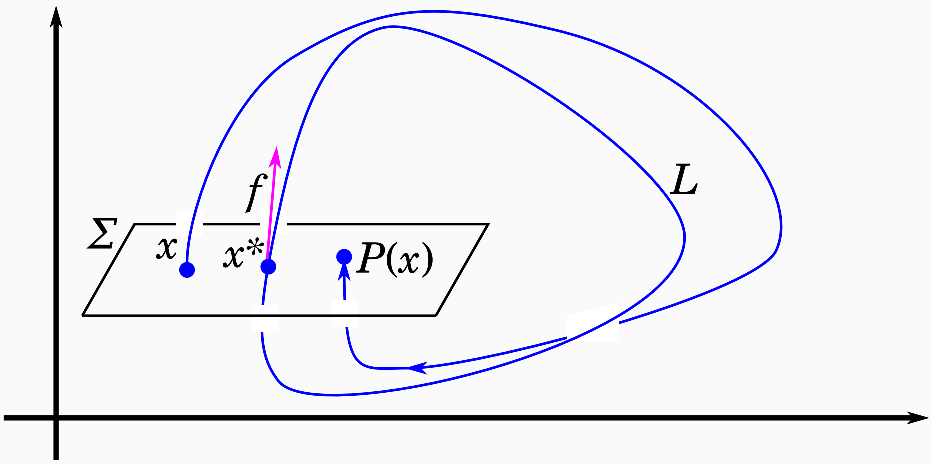
$$x \to S_{T_0}(x) = \Phi^{T_0}(x);$$

this is called the T_0 shift map, and is most useful for non-autonomous systems with time-periodic forcing.



Poincaré map II

For autonomous systems, it is more useful to consider the case when trajectories repeatedly return to a region of phase space, for example, near a periodic orbit L

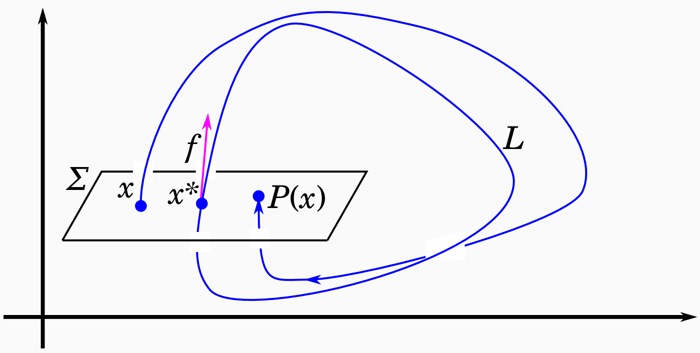


In this context, the Poincaré map is sometimes called the first return map.

Poincaré map III

Recall, we are solving $\frac{dx}{dt} = f(x)$. Suppose that there is a periodic orbit L with period T_L, and that there are no equilibria near L. Then the vector field f is tangent to L at every point on L, and $|f| > 0$ in a neighbourhood of L. Take a point $x^* \in L$ and a cross-section Σ, which is an $(n-1)$-dimensional plane containing x^* and orthogonal to $f(x^*)$, restricted to a neighbourhood of L.

Note: Σ is called a codimension-one surface, since it is specified by one equation: $(x - x^*) \cdot f(x^*) = 0$.



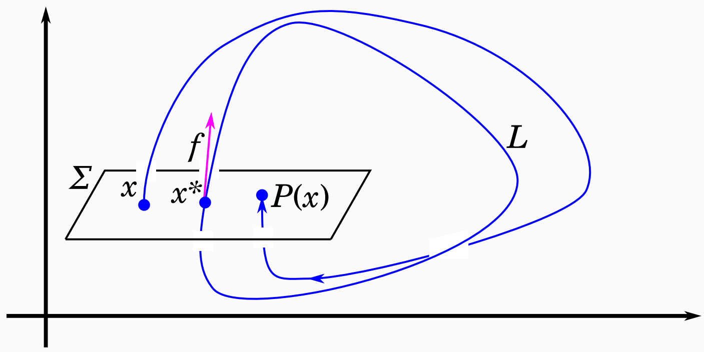
Poincaré map IV

Now take an initial condition $x \in \Sigma$, with x close to x^*. By continuity with respect to initial conditions, there is a time $T > 0$ close to T_L such that $\Phi^T(x) \in \Sigma$. Define the Poincaré map:

$$x \to P(x) = \Phi^{T(x)}(x).$$

This map P is a map $\mathbb{R}^n \to \mathbb{R}^n$ and also a map $\Sigma \to \Sigma$.

Note that x^* is a fixed point of P, that is, $x^* = P(x^*)$.



Linearisation I

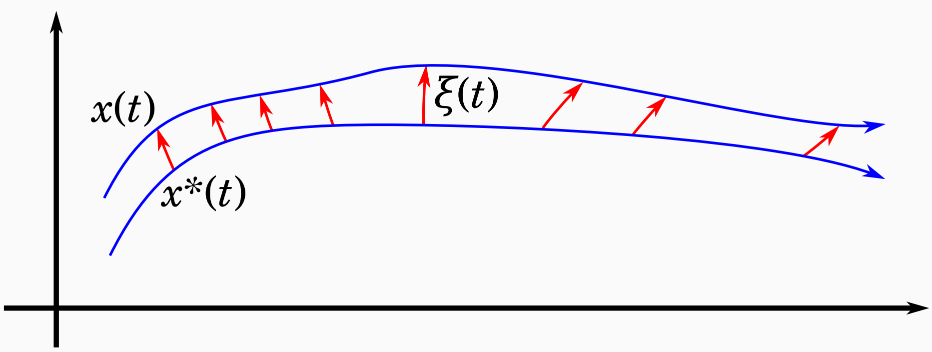
We can establish stability of equilibrium points and periodic orbits by linearising the system.

Consider a (smooth) ODE

$$\dot{x} = f(x)$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $x^*(t)$ be a particular solution of this ODE. Choose an initial condition x_0 close to $x^*(0)$, solve the ODE to find $x(t)$, and define $\xi = x(t) - x^*(t)$. Then ξ satisfies

$$\dot{\xi} = f(x^*(t) + \xi) - f(x^*(t)).$$



Linearisation II

Expand this equation in a Taylor series:
$f(x^*(t) + \xi) = f(x^*(t)) + J(t)\xi + \dots$, where

$$J(t) = \left.\frac{\partial f}{\partial x}\right|_{x=x^*(t)}$$

is called the Jacobian and is an $n \times n$ time-dependent matrix. For infinitesimal ξ, we get:

$$\dot{\xi} = J(t)\xi.$$

The space of initial perturbations from $x^*(0)$ is spanned by n vectors. We can solve for all of these at once by writing:

$$\dot{M} = J(t)M,$$

where M is an $n \times n$ matrix, with $M(0) =$ Identity. This is called the variational equation (as is sometimes the equation $\dot{\xi} = J(t)\xi$).

Stability of equilibrium points I

If x^* is an equilibrium point (and so doesn't depend on time), then J is the usual constant Jacobian matrix:

$$J = \left.\frac{\partial f}{\partial x}\right|_{x=x^*}$$

ξ obeys a linear constant coefficient equation $\dot{\xi} = J\xi$, and the stability of the equilibrium point is determined by the eigenvalue and eigenvector structure of J.

In general, J can be block diagonalised (put into Jordan normal form) with 1×1 blocks for simple or complex eigenvalues, and with larger blocks when there are repeated eigenvalues, for example:

$$J = \left[\begin{array}{c|cc} \lambda_1 & 0 & 0 \\ \hline 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{array}\right]$$

Stability of equilibrium points II

For linear systems, we have:

- If the eigenvalues of J are all distinct and have real parts less than or equal to zero, the equilibrium point is Lyapunov stable.
- If the eigenvalues of J are repeated, the equilibrium point is Lyapunov stable if all the eigenvalues of J have real parts less than or equal to zero, and all eigenvalues on the imaginary axis correspond to 1×1 Jordan blocks; (Exercise: give an example of this.)

We could also make use of the solution of $\dot{\xi} = J\xi$, which is $\xi(t) = \exp(Jt)\xi(0)$, or equivalently, $M(t) = \exp(Jt)$.

More generally, for linear and nonlinear systems, we have that the equilibrium point is stable if all the eigenvalues of J have real parts less than zero.

Stability of equilibrium points III

Example: consider the parameter-dependent ODE:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -x + x(y^2 + z^2) \\ \epsilon y - z - y(y^2 + z^2) - zx^2 \\ y + \epsilon z - z(y^2 + z^2) + yx^2 \end{pmatrix}$$

The Jacobian at any point is:

$$J = \begin{bmatrix} -1 + y^2 + z^2 & 2xy & 2xz \\ -2xz & \epsilon - 3y^2 - z^2 & -1 - 2yz - x^2 \\ 2xy & 1 - 2yz + x^2 & \epsilon - y^2 - 3z^2 \end{bmatrix}$$

The point $(0, 0, 0)$ is an equilibrium point, and $J(0, 0, 0)$ is

$$J(0, 0, 0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \epsilon & -1 \\ 0 & 1 & \epsilon \end{bmatrix}$$

Stability of equilibrium points IV

The eigenvalues of $J(0,0,0)$ are -1 and $\epsilon \pm i$, (all simple eigenvalues), so the equilibrium point is stable if $\epsilon < 0$. In the linearised system, the equilibrium point is Lyapunov stable if $\epsilon \leq 0$.

Exercise: show that for a 2×2 Jacobian matrix (or a 2×2 block, as in this example), the eigenvalues are determined by the Trace and Determinant of the matrix. In particular, the equilibrium point is stable if the Trace is negative and the Determinant is positive.

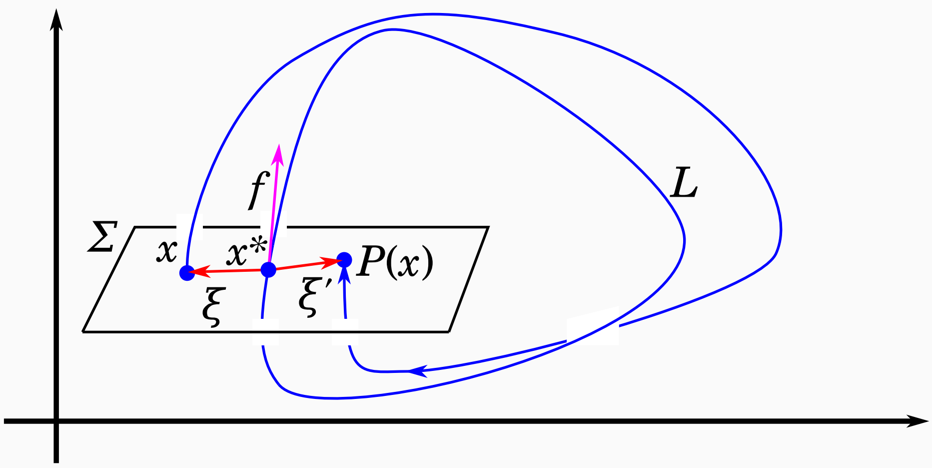
Of course, this does not apply directly to 3×3 or larger Jacobian matrices!

Exercise: If $\mathrm{tr}(J) = \nabla \cdot f < 0$ for all $x(t)$, show that volumes in phase space contract. Systems that satisfy this condition are called dissipative. (In the example on the previous page, $\mathrm{tr}(J) = -1+y^2+z^2+\epsilon-3y^2-z^2+\epsilon-y^2-3z^2 = -1+2\epsilon-3(y^2+z^2)$, so $\mathrm{tr}(J) < 0$ and the system is dissipative as long as $\epsilon < \frac{1}{2}$.)

Stability of periodic orbits I

Suppose L is a periodic orbit with period T_L, and that x^* is a point on L. Take a cross-section $\Sigma = \{x \in X | (x - x^*) \cdot f(x^*) = 0\}$, and define the Poincaré map as before: $x \to P(x) = \Phi^{T(x)}(x)$, where $T(x)$ is the time it takes for the orbit of x to return to Σ, close to x^*.

The map can be linearised: let ξ be the initial perturbation and ξ' the perturbation after going once around the periodic orbit.



Stability of periodic orbits II

We can think of ξ' in terms of the linearised Poincaré map:

$$\xi' = \left.\frac{\partial P}{\partial x}\right|_{x=x^*} \xi$$

but it is also useful to express ξ' in terms of solutions of the variational equation $\dot{M} = J(t)M$, with $M(0) =$ Identity:

$$\xi' = M(T_L)\,\xi;$$

in this context, $M(T_L)$ is called the monodromy matrix of the periodic orbit.

The eigenvalues of $M(T_L)$ are called the Floquet multipliers of the periodic orbit, and these determine its stability: if any are greater than one in magnitude, perturbations will grow, and if they are all less than one in magnitude, perturbations will decay – but...

Stability of periodic orbits III

One of the Floquet multipliers is equal to 1. This can be seen from the fact that $\xi(t) = f(x^*(t))$ is a solution of $\dot{\xi} = J(t)\xi$:

$$\dot{\xi} = \frac{d}{dt} f(x^*(t)) = \left.\frac{\partial f}{\partial x}\right|_{x=x^*(t)} \frac{dx^*}{dt} = J(t) f(x^*) = J(t)\xi$$

This means that, in terms of the Poincaré map $P(x)$, if $\xi = f(x^*)$ (where $x^* = x^*(0) = x^*(T_L)$ is the fixed point of $P(x)$), then $\xi' = f(x^*)$, and hence $f(x^*)$ is an eigenvector of the monodromy matrix $M(T_L)$ with eigenvalue 1.

This Floquet multiplier does not influence the stability of the periodic orbit.

Note that $\xi = f(x^*)$ is an initial condition orthogonal to Σ; the remaining $n-1$ eigenvectors of $M(T_L)$ span Σ.

Stability of periodic orbits IV

The Floquet multipliers of the periodic orbit are therefore $\mu_0 = 1$, μ_1, μ_2, ..., μ_{n-1}, and we have:

- For the linearised system, the periodic orbit is Lyapunov stable if the Floquet multipliers μ_1, ..., μ_{n-1} have magnitudes less than or equal to one, and all Floquet multipliers that are on the unit circle correspond to 1×1 Jordan blocks.
- The periodic orbit is asymptotically stable (and also stable) if the Floquet multipliers μ_1, ..., μ_{n-1} have magnitudes less than one (are inside the unit circle).

In practice, Floquet multipliers have to be determined numerically by solving the joint system:

$$\dot{x} = f(x), \quad x(0) = x^* \in L, \qquad \dot{M} = J(x)M \quad M(0) = \mathrm{Id}$$

(note $J(x)$, not $J(t)$), but there are a small number of cases where they can be calculated by hand (exercise).

Stability of other invariant sets I

Stability of more general invariant sets can be defined using Lyapunov exponents: let $m_i(t)$, $i = 1, \ldots, n$, be the eigenvalues of $M(t)$, ordered such that

$$|m_1(t)| \geq |m_2(t)| \geq \cdots \geq |m_n(t)|$$

and define:

$$\ell_i = \lim_{t\to\infty} \frac{1}{t} \log |m_i(t)|$$

whenever the limit exists. The Lyapunov spectrum is the set $\{\ell_1, \ldots, \ell_n\}$ (real), and this depends on the choice of invariant set (or initial condition); ℓ_1 is the largest of these numbers.

Lyapunov exponents give average rates of contraction (if negative) or expansion (if positive) of orbits near the invariant set (or orbit of the given initial condition), and so the invariant set might be thought of as stable if all of the ℓ_i's are negative ($\ell_1 < 0$).

Stability of other invariant sets II

- For an equilibrium point, we have $M(t) = \exp(Jt)$, so $m_i(t) = \exp(\lambda_i t)$, where the λ_i's are the eigenvalues of the constant matrix J, and the Lyapunov exponents are:

$$\ell_i = \lim_{t\to\infty} \frac{1}{t} \log|\exp(\lambda_i t)| = \mathsf{Re}(\lambda_i).$$

  In this case, the equilibrium point is stable whenever all of the ℓ_i's are negative.
- For a periodic orbit, we can take the limit $t \to \infty$ in steps of the period T_L ($t_j = jT_L$), and so $M(t_j) = M(T_L)^j$ and

$$\ell_i = \lim_{j\to\infty} \frac{1}{jT_L} \log|\mu_i^j| = \frac{1}{T_L} \log|\mu_i|$$

  with the eigenvalues of $M(T_L)$ being the Floquet multipliers μ_i (with $i = 0, \ldots, n-1$ in this case). Similarly, the periodic orbit is stable whenever all of the ℓ_i's are negative ($|\mu_i| < 1$), apart from one $\ell_0 = 0$.

Stability of other invariant sets III

- Similar conclusions are obtained for the stability of tori and other non-chaotic invariant sets.
- The Lyapunov spectrum of a general initial condition is the same as the Lyapunov spectrum of its ω-limit set (that is, we can ignore transients).
- Almost every perturbation from the invariant set grows as $\exp(\ell_1 t)$, where ℓ_1 is the largest Lyapunov exponent (possibly excepting those that are constrained to be zero).
- It is sometimes useful to note the Liouville–Ostrogradski formula:

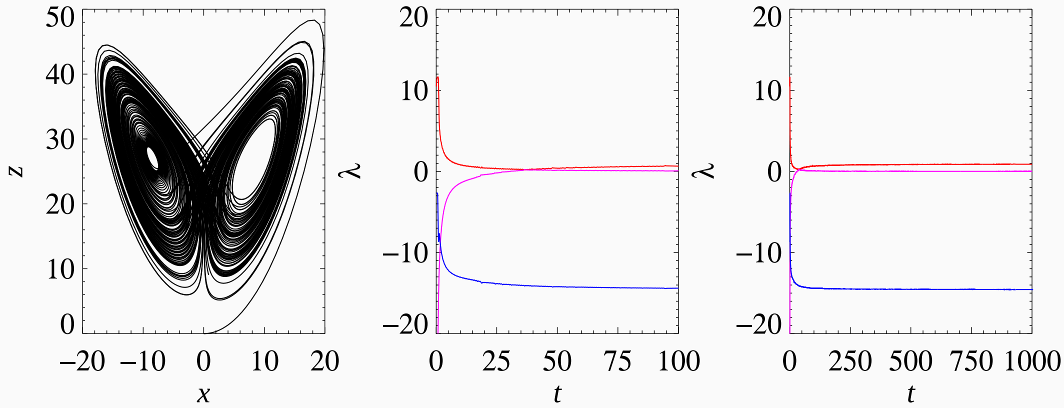
$$\det(M(t)) = \exp\left(\int_0^t \mathrm{tr}(J(t'))\,dt'\right),$$

  and so the sum of the Lyapunov exponents is equal to the average of the trace of $J(t)$:

$$\sum_{i=1}^{n} \ell_i = \lim_{t\to\infty} \frac{\int_0^t \mathrm{tr}(J(t'))\,dt'}{t} = \mathrm{tr}(\bar{J})$$

Stability of other invariant sets IV

- Does $\ell_1 > 0$ imply that the invariant set is not stable? Not necessarily: the Lorenz equations have an unstable "attractor".



$$\dot{x} = \sigma(y - x) \qquad \dot{y} = rx - y - xz \qquad \dot{z} = -bz + xy$$

With $\sigma = 10$, $r = 28$ and $b = 8/3$, we find $l_1 = 0.9043 > 0$, $l_2 = 0.0007 \sim 0$ and $l_3 = -14.5716$. From the L–O formula, we should compare $\mathrm{tr}(J(t)) = \nabla \cdot f = -\sigma - 1 - b = -13.6667$ with $l_1 + l_2 + l_3 = -13.6666$.

Bifurcation theory

Much of Dynamical Systems is concerned with families of ODEs, that is, ODEs that depend on one (or more) parameters $\mu \in \mathbb{R}$ (or $\mu \in \mathbb{R}^m$):

$$\dot{x} = f(x; \mu)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. We would like to understand the qualitative features of the dynamical system:

- How many equilibria are there? Are they stable or unstable?
- How many periodic orbits are there? Are they stable or unstable?
- Are there other kinds of invariant sets: tori or chaotic sets?

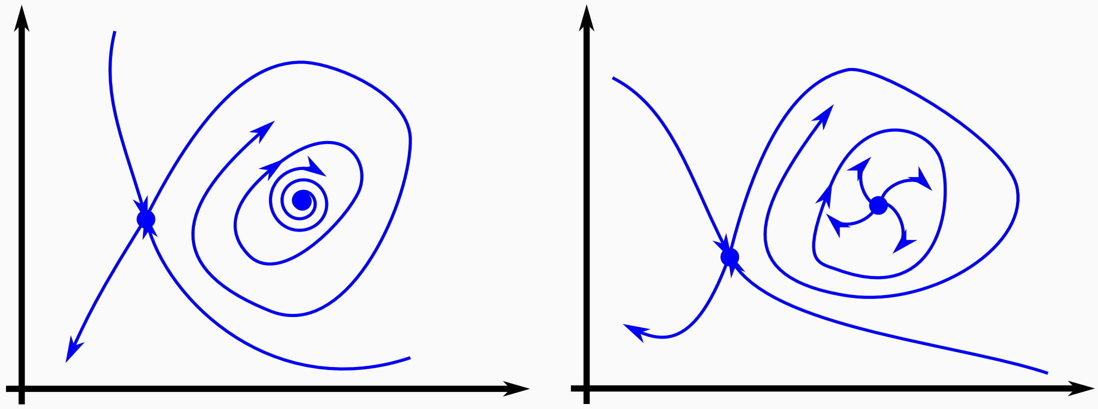
We also want to understand the qualitative changes (or bifurcations) that occur as the parameter μ is changed.

This brings us in to the realm of bifurcation theory. First, we need a definition of a "qualitative change".

Structural stability I

What does it mean to say that two phase portraits (for example, at two different parameter values) are qualitatively the same?

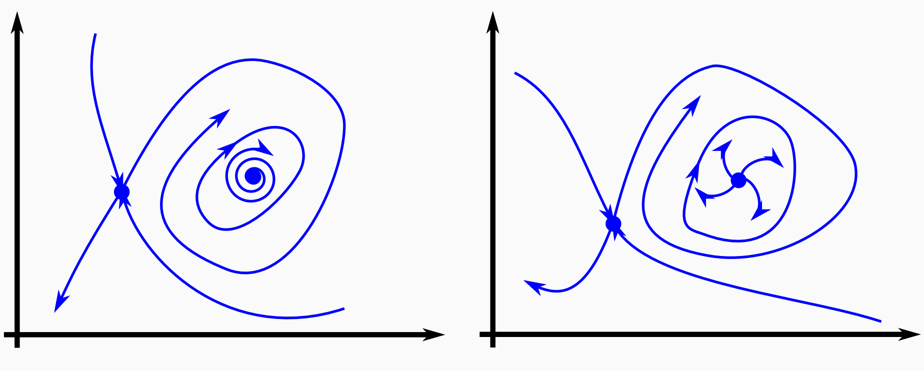
Two dynamical systems in $\mathbb{R}^n$ are topologically equivalent if there is a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ that maps orbits of one onto orbits of the other, preserving the sense of time. (A homeomorphism is an invertible map such that the map and its inverse are continuous.)



Structural stability II

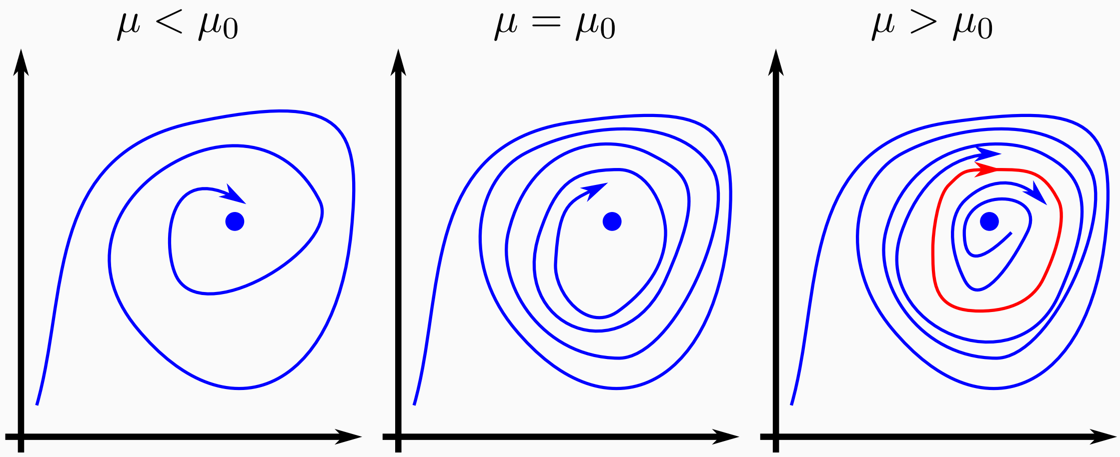
Topological equivalence preserves information about the number, stability and topology of invariant sets, but loses information about transients.

Suppose we consider the family of smooth ODEs $\dot{x} = f(x; \mu)$ for μ close to some particular value μ_0. Then the family is structurally stable at $\mu = \mu_0$ if the phase portrait of $\dot{x} = f(x; \mu)$ is topologically equivalent to the phase portrait of $\dot{x} = f(x; \mu_0)$ whenever μ is sufficiently close to μ_0.



Structural stability III

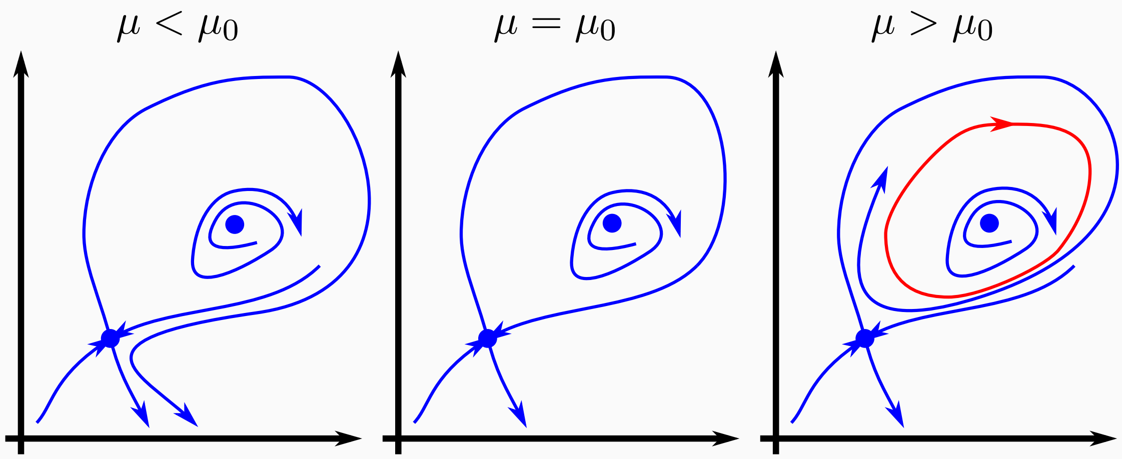
If $\dot{x} = f(x;\mu)$ is not structurally stable at $\mu = \mu_0$ (for example, phase portraits for $\mu > \mu_0$ and for $\mu < \mu_0$ are not topologically equivalent), then we say that a bifurcation has occurred at $\mu = \mu_0$.



Example: a Hopf bifurcation, where an equilibrium point with complex eigenvalues changes stability at $\mu = \mu_0$, and a periodic orbit is created.

Structural stability IV

Example: a global bifurcation, where the stable and unstable manifolds of an equilibrium are rearranged at $\mu = \mu_0$, and a periodic orbit is created.



Structural stability V

How can we tell if two dynamical systems are topologically equivalent? Or conversely, under what conditions are dynamical systems structurally unstable? The two previous examples illustrate the main possibilities, but first we need some definitions.

At an equilibrium point x^* with Jacobian matrix J, let n_-, n_0 and n_+ be the number of eigenvalues of J with real parts less than zero, equal to zero and greater than zero respectively. The equilibrium point is hyperbolic if $n_0 = 0$: if the Jacobian matrix has no eigenvalues on the imaginary axis, and all eigenvectors grow or decay exponentially in time.

Similarly, a periodic orbit is hyperbolic if has no Floquet multipliers on the unit circle (apart from $\mu_0 = 1$). Here, let n_-, n_0 and n_+ be the number of Floquet multipliers (apart from μ_0) inside, on, and outside the unit circle.

Structural stability VI

Theorem (Hartman–Grobman): The phase portraits of two smooth n-dimensional dynamical systems $\dot{x} = f(x)$ and $\dot{y} = g(y)$ near two hyperbolic equilibria x^* and y^* are topologically equivalent iff these equilibria have the same number n_- and n_+ of eigenvalues with negative and positive real parts. (Recall: $n_0 = 0$.)

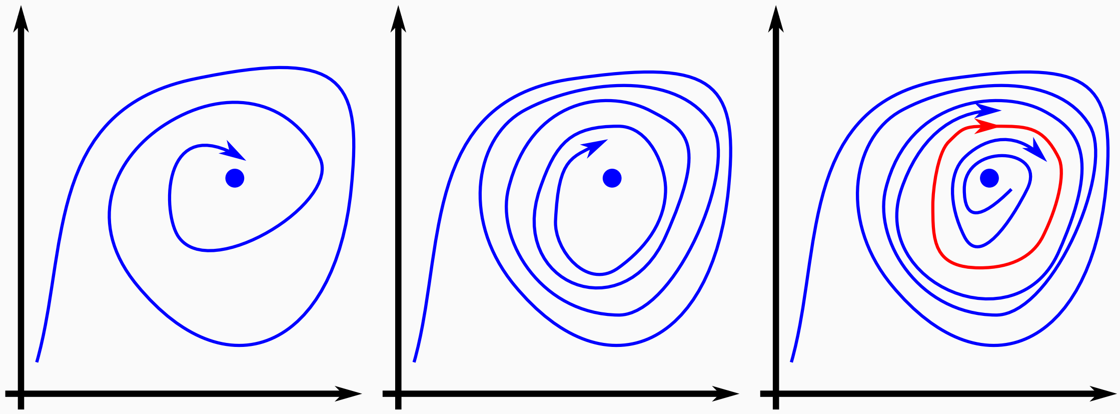
In particular, the nonlinear phase portrait near a hyperbolic equilibrium point is topologically equivalent to the phase portrait of the linearised system.

There is a similar theorem for hyperbolic periodic orbits, with an additional constraint on the sign of the product of the Floquet multipliers inside and outside the unit circle.

So, as the parameter μ is varied in $\dot{x} = f(x; \mu)$, as long as equilibrium points and periodic orbits remain hyperbolic, the dynamical system will remain locally structurally stable (that is, structurally stable near each equilibrium point and periodic orbit).

Structural stability VII

However, a small change to the dynamical system with a nonhyperbolic equilibrium point (or periodic orbit) will perturb eigenvalues on the imaginary axis (or Floquet multipliers on the unit circle) so that their real parts could become positive or negative (magnitudes could become greater or less than one), and so the dynamical system is structurally unstable. In this case, we say that the dynamical system has a local bifurcation.



Structural stability VIII

The second main reason for structural instability is when stable and unstable manifolds intersect nontransversally.

Let x^* be an equilibrium point. We define the linear stable (unstable) manifolds E^s (E^u) to be the spaces spanned by generalised eigenvectors whose eigenvalues have negative (positive) real parts.

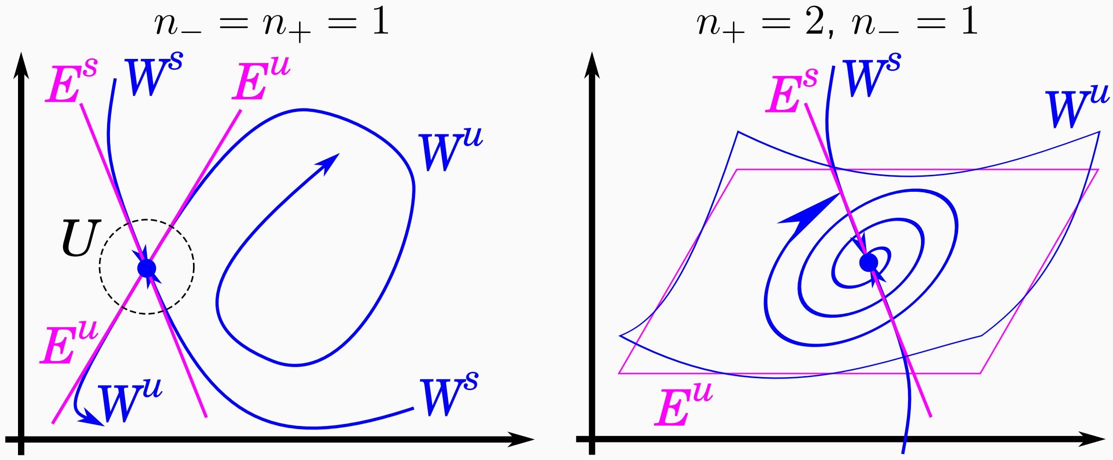
We define the stable (unstable) manifolds W^s (W^u) of x^* to be:

$$W^s(x^*) = \left\{ x \in X \middle| \lim_{t\to\infty} \Phi^t(x) = x^* \right\}$$

with a similar definition of $W^u(x^*)$ but with $t \to -\infty$. Similar definitions can be developed for periodic orbits.

Structural stability IX

Theorem (Hadamard–Perron): If x^* is hyperbolic and we restrict attention to a nbhd U of x^*, then W^s_{loc} and W^u_{loc} are manifolds of dimensions n_- and n_+ respectively, and are tangent to E^s and E^u at x^*. Within these manifolds, trajectories tend towards (or tend away from) x^* at exponential rates.



Structural stability X

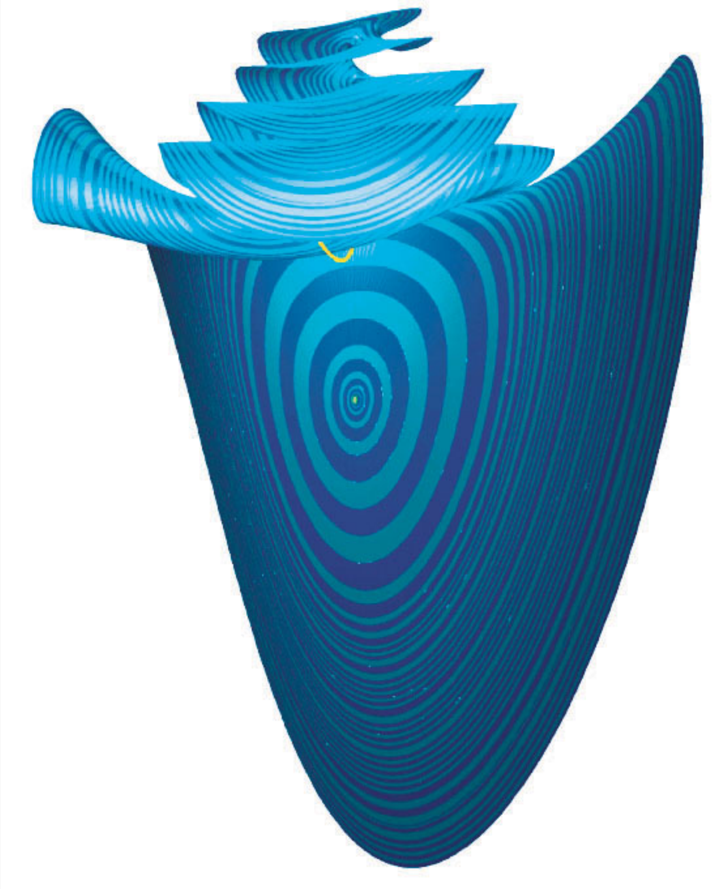
Aside: In our context, an m-dimensional (or codimension-$(n-m)$) invariant manifold (such as W^s and W^u) in $\mathbb{R}^n$ can be thought of as the solution of $n-m$ equations $F_j(x) = 0$, $j = 1, \ldots, n-m$. This implies

$$\frac{dF_j(x(t))}{dt} = \sum_{i=1}^{n} \frac{\partial F_j}{\partial x_i}\frac{dx_i}{dt} = \nabla F_j \cdot f(x) = 0$$

for each j, and for each point x on the manifold (satisfying $F_j(x) = 0$). (Recall $\dot{x}_i = f_i(x)$.)

A method of computing local stable and unstable manifolds can be based on this observation. For methods of computing global manifolds, see Krauskopf, Osinga, Doedel, Henderson, Guckenheimer, Vladimirsky, Dellnitz and Junge, *A survey of methods for computing (un)stable manifolds of vector fields*, Int. J. Bifurcation and Chaos **15** (2005) 763–791, doi:10.1142/S0218127405012533

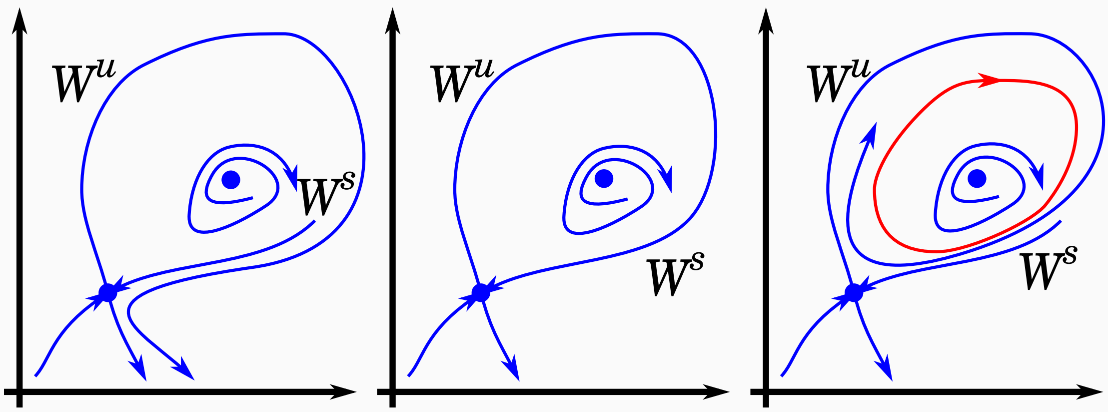
Structural stability XI



The stable manifold of the origin in the Lorenz equations, from Krauskopf and Osinga on DSweb

Structural stability XII

Two manifolds M_1 and M_2 of dimensions m_1 and m_2 in $\mathbb{R}^n$ can intersect transversally if $m_1 + m_2 > n$ (think of two spheres in $\mathbb{R}^3$). Transversal intersections do not disappear if the system is perturbed slightly.

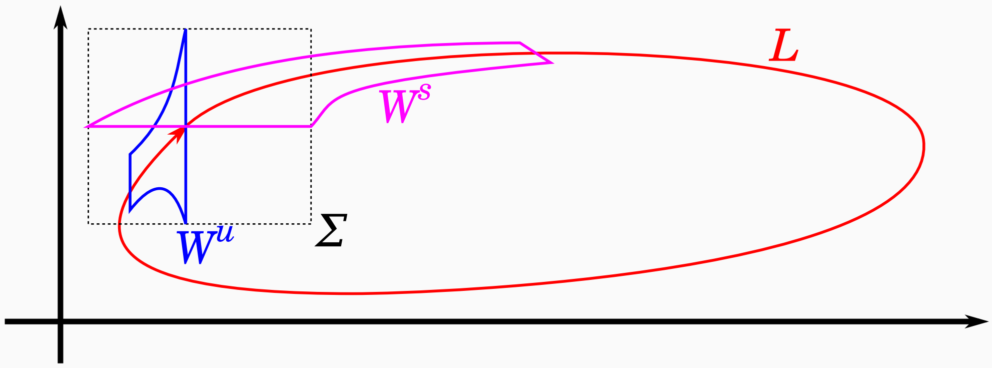


A small perturbation to a system with a nontransversal intersection of stable and unstable manifolds (centre panel) can lead to the creation of periodic orbits. In this case, we say that the system has a global bifurcation.

Structural stability XIII

In summary, a dynamical system has a local bifurcation whenever an equilibrium point or periodic orbit is nonhyperbolic, and it has a global bifurcation whenever stable and unstable manifolds of equilibria or periodic orbits intersect nontransversally.

Global bifurcations are often associated with chaotic dynamics, but complicated dynamics can also be associated with transversal intersections of (for example) the two-dimensional stable and unstable manifolds of a saddle periodic orbit in $\mathbb{R}^3$:



Centre manifold theorem I

A local bifurcation occurs whenever $n_0 \neq 0$ for an equilibrium point or a periodic orbit. In this case, the equilibrium point or periodic orbit has a centre manifold, possibly as well as the stable and unstable manifolds.

Close to the equilibrium point or periodic orbit, the dynamics of the ODE can be thought of as the combination of the dynamics on the stable and unstable manifolds (linear: exponential decay or growth) and the dynamics on the centre manifold (determined by nonlinearities in the ODE). Here we show how to reduce the n-dimensional dynamical system down to an n_0-dimensional dynamical system.

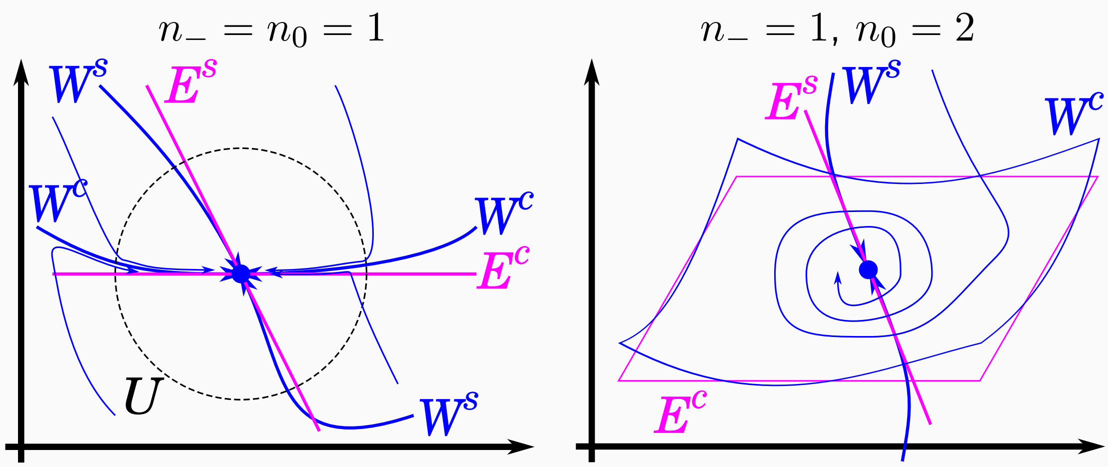
To simplify the presentation, we will concentrate on the case of a nonhyperbolic equilibrium point $x^* = 0$ of the ODE $\dot{x} = f(x)$ (we have removed the parameter for now), and we will assume that $n_+ = 0$ (all eigenvalues of the Jacobian matrix at x^* have negative or zero real part).

Centre manifold theorem II

Theorem (Centre Manifold Theorem): There is a locally defined n_0-dimensional manifold W^c_{loc} that is tangent to E^c at $x = 0$ (where E^c is the space spanned by the eigenvectors whose eigenvalues have zero real part). Moreover, there is a nbhd U of $x = 0$ such that if $\Phi^t(x) \in U$ for all $t \geq 0$, then $\Phi^t(x) \to W^c_{loc}$.

Remarks: W^c_{loc} is called the local centre manifold: it may not be unique (but this turns out not to be important), and it may be less smooth than f. The last sentence implies that all trajectories that stay in U go to W^c_{loc}. We will drop the subscript loc from now on.

Centre manifold theorem III



In these examples (with $n_+ = 0$), trajectories move rapidly (roughly parallel to E^s) towards W^c, and then move slowly on (or close to) W^c.

Centre manifold theorem IV

To perform a reduction of the dynamics onto the centre manifold, we separate the centre from the stable directions. Suppose that we have performed a linear change of coordinates so that we can write $x = (u, v)$ with $u \in \mathbb{R}^{n_0}$ and $v \in \mathbb{R}^{n-n_0}$, so u represents the centre directions and v represents the stable directions. Then we have:

$$\dot{u} = Au + f(u, v),$$
$$\dot{v} = Bv + g(u, v),$$

where the matrices A and B contain all the eigenvalues on and off the imaginary axis respectively, and f and g contain all the nonlinear terms. Effectively, we have block-diagonalised the Jacobian:

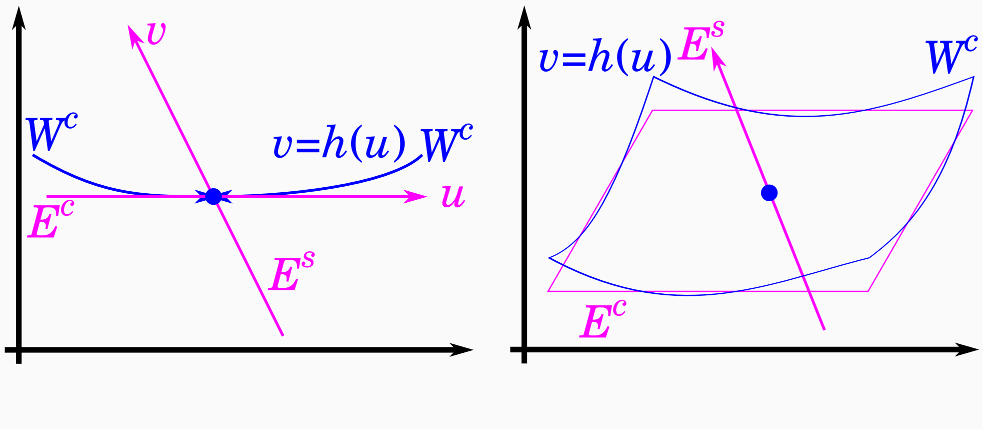
$$J = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$$

Centre manifold theorem V

We could describe the centre manifold in terms of an equation of the form $F(u, v) = 0$ (where $F : \mathbb{R}^{n_0} \times \mathbb{R}^{n-n_0} \to \mathbb{R}^{n-n_0}$), but if we use the fact that E^c is the space $(u, v) = (u, 0)$ and that W^c is tangent to E^c at the equilibrium point $x = 0$, it makes sense to write the centre manifold as:

$$v = h(u), \qquad \text{or equivalently} \quad F(u, v) = h(u) - v = 0,$$

where $h(u) : \mathbb{R}^{n_0} \to \mathbb{R}^{n-n_0}$ and $h(u) = \mathcal{O}(u^2)$ (W^c is tangent to E^c).



Centre manifold theorem VI

We now use the condition that the centre manifold is invariant, that is $\nabla F_j \cdot \dot{x} = 0$, $j = 1, \ldots, n - n_0$, or equivalently

$$\dot{v}_j = \frac{\partial h_j}{\partial u_k} \dot{u}_k, \qquad j = 1, \ldots, n - n_0, \quad \text{or} \quad \dot{v} = \nabla h \cdot \dot{u}$$

We use the ODEs for u and v:

$$Bv + g(u, v) = \nabla h \cdot (Au + f(u, v)).$$

This has to hold for each point on the centre manifold, so we can use $v = h(u)$:

$$Bh(u) + g(u, h(u)) = \nabla h(u) \cdot (Au + f(u, h(u))).$$

Finally, we use a Taylor expansion of this to compute $h(u)$, and end up with the dynamics on the centre manifold:

$$\dot{u} = Ax + f(u, h(u))$$

Centre manifold theorem VII

Theorem (Centre Manifold Reduction Theorem): The ODEs $\dot{x} = f(x)$, having an equilibrium point at $x = 0$, which has n_0 eigenvalues on the imaginary axis, can be written in the form

$$\begin{aligned} \dot{u} &= Au + f(u, v), \\ \dot{v} &= Bv + g(u, v), \end{aligned}$$

where $x = (u, v)$, $u \in \mathbb{R}^{n_0}$ and $v \in \mathbb{R}^{n - n_0}$. These ODEs are locally topologically equivalent to the ODEs

$$\begin{aligned} \dot{u} &= Au + f(u, h(u)), \\ \dot{v} &= Bv, \end{aligned}$$

where $v = h(u)$ describes the local center manifold. These equations are uncoupled, and so the dynamics of the original n-dimensional ODE is effectively reduced to that of an n_0-dimensional ODE.

Computing the centre manifold I

We'll do an example of computing the centre manifold of an ODE, and show how to deal with the disappearing parameter.

Consider the ODE:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \mu x_1 - x_1 x_2 \\ -x_2 + x_1^2 \end{pmatrix}.$$

This has an equilibrium point at $(x_1, x_2) = (0, 0)$. The Jacobian matrix is

$$J = \begin{bmatrix} \mu - x_2 & -x_1 \\ 2x_1 & -1 \end{bmatrix} \qquad \text{so at the eqm point:} \qquad J = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix}.$$

The eigenvalues are μ and -1, so the ODE is structurally unstable when $\mu = 0$ (one eigenvalue is on the imaginary axis), and for this parameter value, the equilibrium point has a centre manifold.

Computing the centre manifold II

Write $x_1 = u$ and $x_2 = v$ to get the ODE into the required form (with $\mu = 0$):

$$\begin{aligned} \dot{u} &= Au + f(u, v) = 0 \times u - uv, \\ \dot{v} &= Bv + g(u, v) = -v + u^2, \end{aligned}$$

so

$$A = \begin{bmatrix} 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \end{bmatrix}, \quad f(u, v) = -uv, \quad g(u, v) = u^2.$$

The linear centre manifold E^c is the space spanned by $\binom{1}{0}$, the eigenvector of 0. The nonlinear centre manifold W^c is tangent to this space at the equilibrium point, so we write

$$v = h(u) = Cu^2 + Du^3 + Eu^4 + \dots,$$

where C, D and E are undetermined coefficients in the Taylor series for h.

Computing the centre manifold III

We now differentiate $v = h(u)$ with respect to time:

$$\dot{v} = \nabla h \cdot \dot{u} = (2Cu + 3Du^2 + 4Eu^3 + \dots)\dot{u}.$$

Substitute the ODEs for u and v:

$$Bv + g(u, v) = -v + u^2 = (2Cu + 3Du^2 + 4Eu^3 + \dots)(0 \times u - uv).$$

Now impose that we are on the centre manifold, so
$v = h(u) = Cu^2 + \dots$:

$$-(Cu^2 + Du^3 + Eu^4 + \dots) + u^2 = (2Cu + 3Du^2 + \dots)(-u)(Cu^2 + Du^3 + \dots).$$

The equation should hold for all values of u, so the coefficients of each power of u on each side should match.

Computing the centre manifold IV

The LHS is:

$$(1 - C)u^2 - Du^3 - Eu^4 + \dots$$

while the RHS is

$$(-2C^2)u^4 + (-5CD)u^5 + (-3D^2 - 6CE)u^6 + \dots,$$

so matching the two sides gives us $C = 1$, $D = 0$, $E = 2$, $\dots$, and an approximate centre manifold:

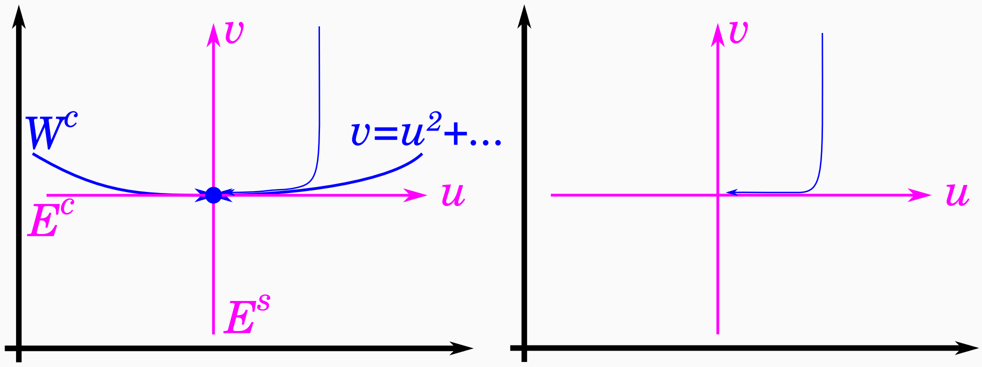
$$v = u^2 + 2u^4 + \dots$$

The dynamics on the centre manifold is given by

$$\dot{u} = -uv = -u^3 - 2u^5 + \dots, \qquad \dot{v} = -v.$$

Note that $u = 0$ is still a non-hyperbolic equilibrium point, but now we know it is nonlinearly stable.

Computing the centre manifold V



Left: trajectories rapidly decay onto the centre manifold W^c, given approximately by $v = u^2 + \dots$.

Right: after the centre manifold reduction, we can explicitly separate the v dynamics (fast decay) from the u dynamics (slow evolution).

Computing the centre manifold VI

However, we have lost the parameter μ (recall, we sent it to zero in order for there to be a center manifold). The parameter can be restored by using an extended centre manifold. Introduce a trivial ODE for the parameter:

$$\begin{aligned}\dot{u} &= \mu u - uv,\\ \dot{\mu} &= 0,\\ \dot{v} &= -v + u^2.\end{aligned}$$

Now the μu term is nonlinear, and the linear parts are:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \end{bmatrix},$$

along with nonlinear parts:

$$f((u,\mu),v) = \begin{pmatrix} \mu u - uv \\ 0 \end{pmatrix}, \quad g((u,\mu),v) = u^2.$$

Computing the centre manifold VII

The linear centre manifold for this problem in $\mathbb{R}^3$ is the (u, μ) plane; the nonlinear centre manifold $v = h(u, \mu)$ is tangent to this at the origin, so we write:

$$v = C_{20}u^2 + C_{11}u\mu + C_{02}\mu^2 + D_{30}u^3 + D_{21}u^2\mu + D_{12}u\mu^2 + D_{03}\mu^3 + \dots$$

As before, we differentiate this with respect to time ($\dot{v} = \nabla h \cdot \dot{u}$):

$$\dot{v} = \left(2C_{20}u + C_{11}\mu + 3D_{30}u^2 + 2D_{21}u\mu + D_{12}\mu^2 + \dots\right)\dot{u} + (\dots)\,\dot{\mu},$$

substitute in the ODEs for u, μ and v:

$$-v + u^2 = \left(2C_{20}u + C_{11}\mu + 3D_{30}u^2 + 2D_{21}u\mu + D_{12}\mu^2 + \dots\right)(\mu u - uv),$$

impose that we are on the centre manifold:

$$-(C_{20}u^2 + C_{11}u\mu + C_{02}\mu^2 + \dots) + u^2 = (2C_{20}u + C_{11}\mu + \dots)\,(\mu u - u(C_{20}u^2 + C$$

Computing the centre manifold VIII

and match coefficients of terms in u and μ to get

$$C_{20} = 1, \quad D_{21} = -2, \quad C_{11} = C_{02} = D_{30} = D_{12} = D_{03} = 0.$$

This results in an expression for the extended centre manifold:

$$v = u^2 - 2u^2\mu + \dots$$

and so the dynamics on the extended centre manifold is:

$$\dot{u} = \mu u - (1 - 2\mu)u^3 + \dots, \qquad \dot{\mu} = 0.$$

Finally, we restore the interpretation of μ as a parameter and obtain:

$$\dot{u} = \mu u - (1 - 2\mu)u^3 + \dots$$

We would have obtained the first term either by going only to second order in $h(u, \mu)$, or indeed by just sticking the μu term back in.

Examples of local bifurcations of equilibria I

Suppose we started with an n-dimensional ODE $\dot{x} = f(x; \mu)$ and determined that an equilibrium point $x = x^*$ has a local bifurcation at parameter value $\mu = \mu^*$, with n_0 eigenvalues on the imaginary axis. This implies that x^* has an n_0-dimensional centre manifold, and that we can perform a centre manifold reduction from the n-dimensional ODE down to an n_0-dimensional ODE of the form

$$\dot{u} = g(u; \nu),$$

where $u = 0$ is an equilibrium point that has n_0 eigenvalues on the imaginary axis when $\nu = 0$, so the system is structurally unstable, and has a local bifurcation, at $\nu = 0$. The reduction is valid for small u and ν, that is, for x close to x^* and for μ close to μ^*.

The simplest (codimension-one) ways that $u = 0$ can have eigenvalues on the imaginary axis is if the eigenvalue is zero ($n_0 = 1$), or there is a pair of pure imaginary eigenvalues ($n_0 = 2$).

Examples of local bifurcations of equilibria II

With a zero eigenvalue, we expand $\dot{u} = g(u; \nu)$ in a Taylor series:

$$\dot{u} = g_{00} + g_{10}u + g_{01}\nu + g_{20}u^2 + g_{11}u\nu + g_{02}\nu^2 + g_{30}u^3 + \dots$$

We know that when $\nu = 0$, $u = 0$ is an equilibrium point:

$$g(0; 0) = g_{00} = 0.$$

We also know that when $\nu = 0$, $u = 0$ is non-hyperbolic (the eigenvalue is zero):

$$J(0; 0) = \frac{\partial g}{\partial u}(0; 0) = g_{10} = 0.$$

This implies that for small u and ν, the ODE is given by:

$$\dot{u} = g_{01}\nu + g_{20}u^2 + g_{11}u\nu + g_{02}\nu^2 + g_{30}u^3 + \dots,$$

where generically all the coefficients in the expansion are non-zero.

Examples of local bifurcations of equilibria III

We would like to simplify this – and we will aim for the normal form of a saddle-node bifurcation:

$$\dot{z} = \pm\lambda \pm z^2,$$

by performing a coordinate transformation from $(u, \nu) \to (z, \lambda)$. There are two ways to acheive this: we can appeal to order of magnitude estimates, or we can construct the explicit coordinate transformation (exercise).

Order of magnitude estimates: we know that u and ν are both small ($u \ll 1$ and $\nu \ll 1$), so $u^3 \ll u^2$ and $\nu^2 \ll \nu$, so we will drop the $g_{30}u^3$ and $g_{02}\nu^2$ terms. All higher order terms in the $\ldots$ can be dropped by similar comparisons.

We need to keep the $g_{20}u^2$ term (else the equation would not be nonlinear), and at least one of the $g_{01}\nu$ and $g_{11}u\nu$ terms (else there would be no parameter), and all the terms that we keep

Examples of local bifurcations of equilibria IV

should be of the same magnitude (else we'd end up with only one term). If $u^2 \sim u\nu$, then $u \sim \nu$ and the $g_{01}\nu$ term is larger than the other two – not allowed. If on the other hand $u^2 \sim \nu$, then $u\nu \sim u^3 \ll u^2$, so we can drop $g_{11}u\nu$. We thus end up with

$$\dot{u} = g_{01}\nu + g_{20}u^2 + \ldots$$

We now define $z = u/|g_{20}|$ and $\lambda = \nu/|g_{20}g_{01}|$, and obtain

$$\dot{z} = \pm\lambda \pm z^2 + \ldots,$$

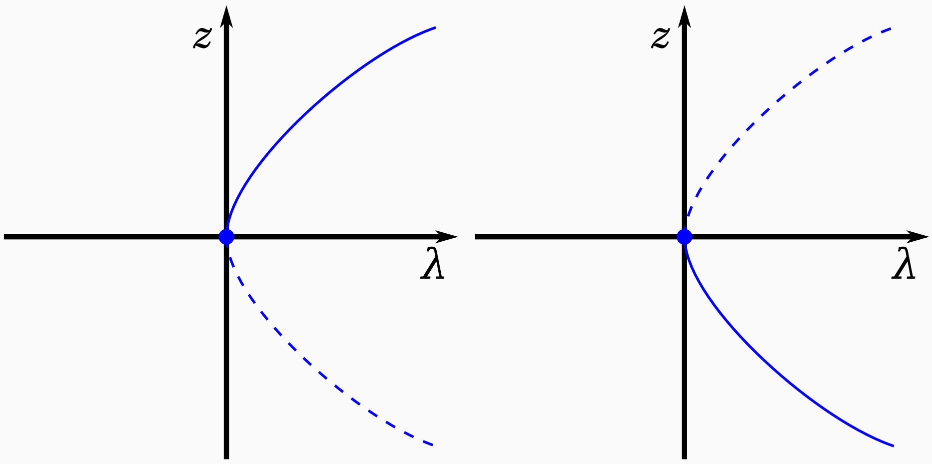
where the two signs $\pm\lambda$ and $\pm z^2$ are determined by the signs of g_{01} and g_{20}. The last (and most dodgy) step is to assert that the $\ldots$ do not change the picture and so can be removed, and we get:

The saddle-node normal form: $\quad \dot{z} = \pm\lambda \pm z^2.$

Clearly this derivation fails if either $g_{01} = 0$ or $g_{20} = 0$.

Examples of local bifurcations of equilibria V

Taking $\dot{z} = \lambda - z^2$ to be definite, the equilibria are $z = \pm\sqrt{\lambda}$, and their stability is given by $J = \frac{\partial \dot{z}}{\partial z} = -2z$, so the eqm point with positive z is stable, the other is unstable (see below). This is called a saddle-node bifurcation; it is also known as a fold bifurcation or a limit point. In these bifurcation diagrams, solid (dashed) lines represent stable (unstable) equilibria.



$\dot{z} = \lambda - z^2$ $\quad\quad$ $\dot{z} = -\lambda + z^2$

Examples of local bifurcations of equilibria VI

Theorem (Saddle-node bifurcation theorem): The one-dimensional ODE $\dot{u} = g(u; \nu)$, having at $\nu = 0$ an equilibrium point $u = 0$ with eigenvalue $g_u(0; 0) = 0$, and satisfying the non-degeneracy conditions:

- $g_{uu}(0; 0) \neq 0$ (that is, $g_{20} \neq 0$)
- $g_\nu(0; 0) \neq 0$ (that is, $g_{01} \neq 0$)

is locally topologically equivalent to one of the following normal forms:

$$\dot{z} = \pm\lambda \pm z^2.$$

A one-dimensional ODE with a non-hyperbolic equilibrium point satisfying the two non-degeneracy conditions is said to be generic.

Examples of local bifurcations of equilibria VII

There are reasons that a particular ODE might be non-generic:

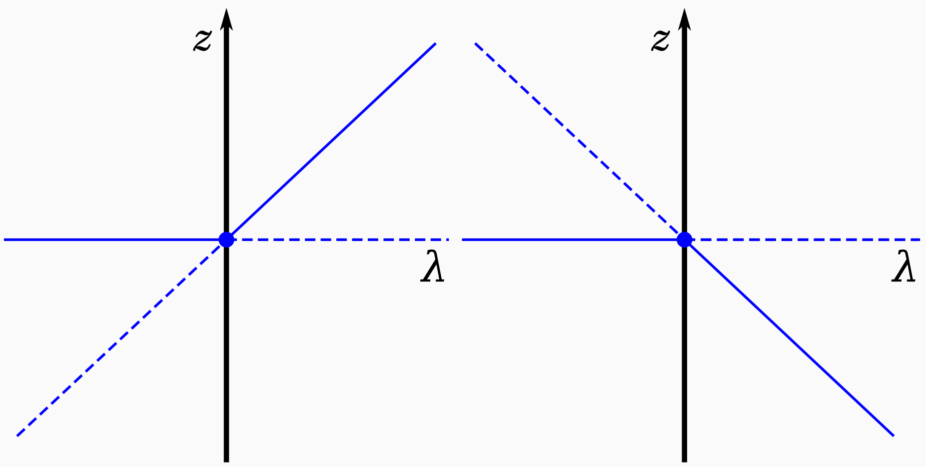
- On varying a second parameter, one or other of the non-degeneracy conditions fails. For example, consider

$$\dot{u} = \nu - Au^2 + \ldots;$$

  when $A = 0$, some of the higher order terms must be retained. This is called a codimension-two bifurcation.
- There is a modelling reason for the non-degeneracy conditions to fail. For example, if a population reaches zero individuals, then it must remain zero for all future times and for all parameter values. This implies $g(0;\nu) = 0$, and so $g_\nu(0;0) = 0$. This will lead to the transcritical bifurcation.
- The system has a symmetry, so that $g(-u;\nu) = -g(u;\nu)$ for all u and ν. This implies that $g_\nu(0;0) = 0$ and $g_{uu}(0;0) = 0$. This will lead to the pitchfork bifurcation.

Examples of local bifurcations of equilibria VIII

Transcritical bifurcation: take $\dot{z} = \lambda z - z^2$ as an example. The equilibria are $z = 0$ and $z = \lambda$; their stability is given by $J = \lambda - 2z$, so $J(0) = \lambda$ ($z = 0$ is stable for $\lambda < 0$), and $J(\lambda) = -\lambda$ ($z = \lambda$ is stable for $\lambda > 0$).



$\dot{z} = \lambda z - z^2$ $\dot{z} = \lambda z + z^2$

Examples of local bifurcations of equilibria IX

Theorem (transcritical bifurcation theorem): The one-dimensional ODE $\dot{u} = g(u;\nu)$, having at $u = 0$ as an equilibrium point for all ν ($g(0;\nu) = 0$), this equilibrium point having a zero eigenvalue at $\nu = 0$ ($g_u(0;0) = 0$), and satisfying the non-degeneracy conditions:

- $g_{uu}(0;0) \neq 0$ (that is, $g_{20} \neq 0$)
- $g_{u\nu}(0;0) \neq 0$ (that is, $g_{11} \neq 0$)

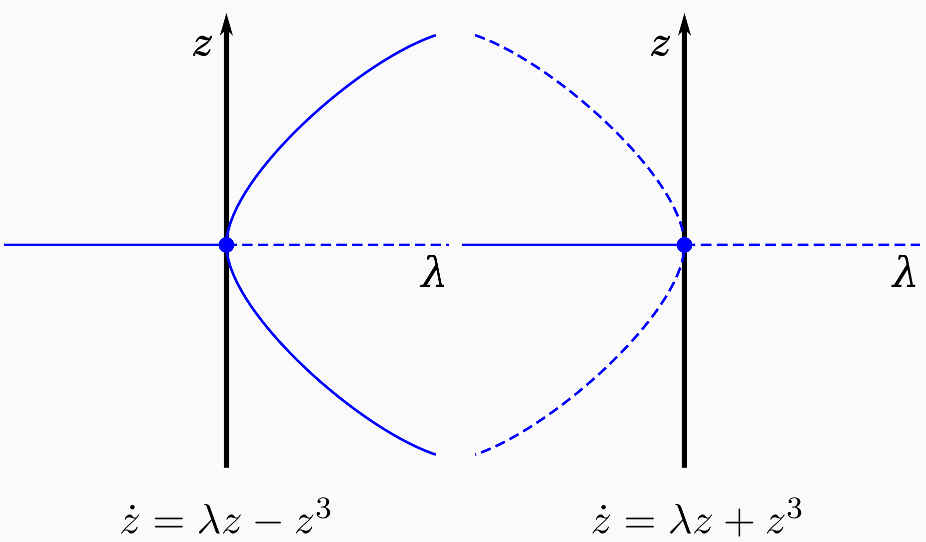
is locally topologically equivalent to one of the following normal forms:

$$\dot{z} = \pm\lambda z \pm z^2.$$

The transcritical bifurcation is sometimes also known as exchange of stabilities.

Examples of local bifurcations of equilibria X

Pitchfork bifurcation: take $\dot{z} = \lambda z - z^3$ as an example. The equilibria are $z = 0$ and $z = \pm\sqrt{\lambda}$ ($\lambda \geq 0$); their stability is given by $J = \lambda - 3z^2$, so $J(0) = \lambda$ ($z = 0$ is stable for $\lambda < 0$), and $J(\pm\sqrt{\lambda}) = -2\lambda$ ($z = \pm\sqrt{\lambda}$ is stable for $\lambda > 0$).



Examples of local bifurcations of equilibria XI

Theorem (pitchfork bifurcation theorem): The one-dimensional ODE $\dot{u} = g(u;\nu)$, having a symmetry such that $g(-u;\nu) = -g(u;\nu)$, which implies that $u = 0$ is an equilibrium point for all ν, this equilibrium point having a zero eigenvalue at $\nu = 0$ ($g_u(0;0) = 0$), and satisfying the non-degeneracy conditions:

- $g_{uuu}(0;0) \neq 0$ (that is, $g_{30} \neq 0$)
- $g_{u\nu}(0;0) \neq 0$ (that is, $g_{11} \neq 0$)

is locally topologically equivalent to one of the following normal forms:

$$\dot{z} = \pm\lambda z \pm z^3.$$

If the normal form is $\dot{z} = \pm\lambda z - z^3$, the pitchfork bifurcation is said to be supercritical, otherwise, if the normal form is $\dot{z} = \pm\lambda z + z^3$, the pitchfork bifurcation is said to be subcritical.

Examples of local bifurcations of equilibria XII

These are examples of bifurcation diagrams, in which bifurcations, which separate parameter regions with topologically equivalent phase portraits, are identified (and representative phase portraits are sketched).

Note: the transcritical and pitchfork bifurcations are themselves structurally unstable, in that they are destroyed if a general perturbation is added to the normal form:

$$\dot{z} = \kappa + \lambda z - z^2$$

or

$$\dot{z} = \kappa + \lambda z - z^3$$

(Exercise).

Examples of local bifurcations of equilibria XIII

The last example of a codimension-one local bifurcation is the Hopf bifurcation (or Andronov–Hopf bifurcation), which occurs whenever an equilibrium point has pure imaginary eigenvalues.

Consider for example the pair of ODEs

$$\begin{aligned}\dot{x} &= \mu x - \omega y - Ax(x^2+y^2) - By(x^2+y^2),\\ \dot{y} &= \omega x + \mu y - Ay(x^2+y^2) + Bx(x^2+y^2).\end{aligned}$$

With ω and A non-zero, this is called the normal form of the Hopf bifurcation (we will see why later). Then $(x, y) = (0, 0)$ is an equilibrium point; its stability is determined by the Jacobian matrix

$$\begin{aligned}J &= \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix}\\ &= \begin{bmatrix} \mu - 3Ax^2 - Ay^2 - 2Bxy & -\omega - 2Axy - Bx^2 - 3By^2 \\ \omega - 2Axy + 3Bx^2 + By^2 & \mu - Ax^2 - 3Ay^2 - 2Bxy \end{bmatrix}\end{aligned}$$

Examples of local bifurcations of equilibria XIV

so

$$J(0,0) = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}.$$

The eigenvalues of this matrix are $\mu \pm i\omega$, so the equilibrium point $(x, y) = (0, 0)$ is non-hyperbolic when $\mu = 0$ (eigenvalues are pure imaginary), and there is a Hopf bifurcation at this parameter value.

The easiest way to see what happens for μ close to zero is to go in to polar coordinates. Let

$$x(t) = r(t)\cos\theta(t) \qquad \text{and} \qquad y(t) = r(t)\sin\theta(t),$$

or equivalently,

$$r^2(t) = x^2(t) + y^2(t) \qquad \text{and} \qquad \theta(t) = \tan^{-1}(y(t)/x(t)).$$

Examples of local bifurcations of equilibria XV

Then

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \qquad \text{and} \qquad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta,$$

or equivalently,

$$r\dot{r} = x\dot{x} + y\dot{y} \qquad \text{and} \qquad r^2\dot{\theta} = x\dot{y} - y\dot{x}.$$

In this case, we have

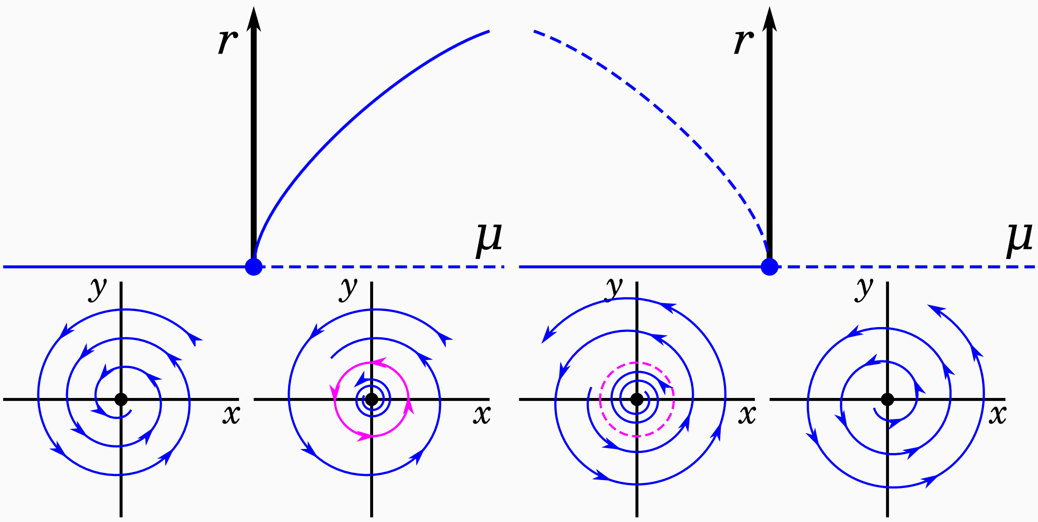
$$\dot{r} = \mu r - Ar^3 \qquad \text{and} \qquad \dot{\theta} = \omega + Br^2.$$

This is the normal form of the Hopf bifurcation in polar coordinates. Since $\omega \neq 0$, there are no equilibria close to $r = 0$ apart from $r = 0$. However, we can identify periodic orbits:

$$r = \sqrt{\mu/A} \qquad \text{and} \qquad \dot{\theta} = \omega + \mu B/A.$$

Examples of local bifurcations of equilibria XVI

Since the two variables r and θ have decoupled, and since the θ direction will contain the Floquet multiplier at unity, the stability of the orbit is determined by the behviour in the r direction, which is the same as the pitchfork:



$$\dot{r} = \mu r - r^3 \qquad\qquad \dot{r} = \mu r + r^3$$

Examples of local bifurcations of equilibria XVII

What do we do if there is a Hopf bifurcation but the ODE is not in the nice normal form? For example, consider

$$\dot{x} = y, \qquad \dot{y} = -\lambda x + \kappa y + Mx^3 + Nx^2 y$$

where M, N and λ are considered to be fixed numbers, and κ is the parameter. There is an equilibrium point $(x, y) = (0, 0)$ (there are others). The Jacobian matrix is

$$J = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\lambda + 3Mx^2 + 2Nxy & \kappa + Nx^2 \end{bmatrix}$$

$$\text{so} \qquad J(0,0) = \begin{bmatrix} 0 & 1 \\ -\lambda & \kappa \end{bmatrix}.$$

The trace and determinant of this matrix are κ and λ respectively, so the eigenvalues are pure imaginary when $\kappa = 0$, provided that $\lambda > 0$ – there is a Hopf bifurcation at $\kappa = 0$. However, there is no easy transformation into polar coordinates...

Birkhoff normal form I

In this example, we take $\kappa = 0$ and $\lambda = 1$ and aim to convert

$$\dot{x} = y, \qquad \dot{y} = -x + Mx^3 + Nx^2 y$$

into the Hopf normal form by carrying out a near-identity transformation: write

$$\begin{aligned} x &= u + P(u, v) \\ y &= v + Q(u, v), \end{aligned}$$

where $P(u, v)$ and $Q(u, v)$ are homogeneous cubic polynomials. This is a near-identity transformation because, for small enough u and v, we have $x \approx u$ and $y \approx v$. The aim is to choose $P(u, v)$ and $Q(u, v)$ so that the ODE in the new (u, v) coordinates matches the Hopf normal form.

Birkhoff normal form II

Since $P(u,v)$ and $Q(u,v)$ are cubic polynomials, we can write the transformation as

$$
\begin{aligned}
x &= u + P_{30}u^3 + P_{21}u^2v + P_{12}uv^2 + P_{03}v^3 \\
y &= v + Q_{30}u^3 + Q_{21}u^2v + Q_{12}uv^2 + Q_{03}v^3.
\end{aligned}
$$

We can also write the inverse transformation approximately as:

$$
\begin{aligned}
u &= x - P(u,v) = x - P(x - P(u,v), y - Q(u,v)) = \dots \\
v &= y - Q(u,v) = y - Q(x - P(u,v), y - Q(u,v)) = \dots,
\end{aligned}
$$

or

$$
\begin{aligned}
u &= x - P(x,y) + \mathcal{O}(4) \\
v &= y - Q(x,y) + \mathcal{O}(4),
\end{aligned}
$$

where we can use $P(x,y) = P_{30}x^3 + P_{21}x^2y + P_{12}xy^2 + P_{03}y^3$ etc.

Birkhoff normal form III

Now we differentiate to get $(\dot{u}, \dot{v})$:

$$
\begin{aligned}
\dot{u} &= \dot{x} - (3P_{30}x^2 + 2P_{21}xy + P_{12}y^2)\dot{x} \\
&\qquad - (P_{21}x^2 + 2P_{12}xy + 3P_{03}y^2)\dot{y} + \mathcal{O}(4) \\
\dot{v} &= \dot{y} - \frac{\partial Q}{\partial x}\dot{x} - \frac{\partial Q}{\partial y}\dot{y} + \mathcal{O}(4).
\end{aligned}
$$

Substitute in the expressions for $(\dot{x}, \dot{y})$:

$$
\begin{aligned}
\dot{u} &= y - (3P_{30}x^2 + 2P_{21}xy + P_{12}y^2)y \\
&\qquad - (P_{21}x^2 + 2P_{12}xy + 3P_{03}y^2)(-x + Mx^3 + Nx^2y) + \mathcal{O}(4) \\
\dot{v} &= -x + Mx^3 + Nx^2y - (3Q_{30}x^2 + 2Q_{21}xy + Q_{12}y^2)y \\
&\qquad - (Q_{21}x^2 + 2Q_{12}xy + 3Q_{03}y^2)(-x + Mx^3 + Nx^2y) + \mathcal{O}(4).
\end{aligned}
$$

Birkhoff normal form IV

Substitute in the expressions for $(x, y) = (u, v) + \dots$:

$$\begin{aligned}
\dot{u} = v &+ Q_{30}u^3 + Q_{21}u^2v + Q_{12}uv^2 + Q_{03}v^3 \\
&- (3P_{30}u^2 + 2P_{21}uv + P_{12}v^2)v \\
&- (P_{21}u^2 + 2P_{12}uv + 3P_{03}v^2)(-u + Mu^3 + Nu^2v) + \mathcal{O}(4) \\
\dot{v} = -&(u + P_{30}u^3 + P_{21}u^2v + P_{12}uv^2 + P_{03}v^3) + Mu^3 + Nu^2v \\
&- (3Q_{30}u^2 + 2Q_{21}uv + Q_{12}v^2)v \\
&- (Q_{21}u^2 + 2Q_{12}uv + 3Q_{03}v^2)(-u + Mu^3 + Nu^2v) + \mathcal{O}(4),
\end{aligned}$$

or

$$\begin{aligned}
\dot{u} = v &+ (P_{21} + Q_{30})u^3 + (2P_{12} - 3P_{30} + Q_{21})u^2v \\
&+ (3P_{03} - 2P_{21} + Q_{12})uv^2 + (-P_{12} + Q_{03})v^3 + \mathcal{O}(4) \\
\dot{v} = -u &+ (M - P_{30} + Q_{21})u^3 + (N - P_{21} + 2Q_{12} - 3Q_{30})u^2v \\
&+ (-P_{12} + 3Q_{03} - 2Q_{21})uv^2 + (-P_{03} - Q_{12})v^3) + \mathcal{O}(4),
\end{aligned}$$

Birkhoff normal form V

We are aiming for the Hopf normal form:

$$\begin{aligned}
\dot{u} &= v - Au(u^2 + v^2) - Bv(u^2 + v^2) + \mathcal{O}(4) \\
\dot{v} &= -u - Av(u^2 + v^2) + Bu(u^2 + v^2) + \mathcal{O}(4),
\end{aligned}$$

so we set

$$\begin{aligned}
-A &= P_{21} + Q_{30} = 3P_{03} - 2P_{21} + Q_{12} \\
&= N - P_{21} + 2Q_{12} - 3Q_{30} = -P_{03} - Q_{12} \\
B &= -(2P_{12} - 3P_{30} + Q_{21}) = -(-P_{12} + Q_{03}) \\
&= M - P_{30} + Q_{21} = -P_{12} + 3Q_{03} - 2Q_{21}
\end{aligned}$$

This is 8 linear equations for 10 unknowns: A, B and $P_{30}, \dots, Q_{03}$. We can proceed as follows: solve the first pair of equations not involving A by setting

$$Q_{30} = 3P_{03} - 3P_{21} + Q_{12}$$

Birkhoff normal form VI

After further manipulations (try this yourselves), we get a two-parameter family of solutions:

$$P_{30} = \frac{M}{4} + Q_{03}, \quad P_{12} = \frac{3M}{8} + Q_{03}, \quad P_{03} = \frac{N}{8} + P_{21},$$
$$Q_{30} = \frac{N}{8} - P_{21}, \quad Q_{21} = -\frac{3M}{8} + Q_{03}, \quad Q_{12} = -\frac{N}{4} - P_{21},$$

(a two-parameter family of solutions) and

$$A = -\frac{N}{8}, \qquad B = \frac{3M}{8},$$

so the Hopf bifurcation is supercritical when $N < 0$:

$$\dot{r} = \mu r + \frac{N}{8} r^3 + \mathcal{O}(4) \qquad \text{and} \qquad \dot{\theta} = \omega + \frac{3M}{8} r^2 + \mathcal{O}(3).$$

Note that A and B do not depend on the arbitrary parameters in the two-parameter family of solutions. Note also that we have assumed that the $\mathcal{O}(4)$ terms can be dropped.

Birkhoff normal form VII

We consider an n-dimensional ODE with an equilibrium point.

First, we transform the ODE so that the equilibrium point is at the origin, so:

$$\dot{x} = f(x),$$

with $x \in \mathbb{R}^n$ and $f(0) = 0$. We are not writing the parameter for now, and we are interested in the dynamics of the ODE for small x.

Second, we perform a linear change of coordinates so that the linear part of $f(x)$ (the Jacobian matrix) is in Jordan normal form. This linear change of coordinates uses the eigenvectors of J; in the case of repeated eigenvalues, we use generalised eigenvectors, and in the case of complex eigenvalues, we use the real and imaginary parts of the eigenvectors. Thus the first non-zero term in the Taylor series of $f(x)$ is the linear part:

$$\dot{x} = f(x) = Jx + \dots,$$

where J is an $n \times n$ matrix in Jordan normal form.

Birkhoff normal form VIII

It is easiest to proceed if we consider the case where J is diagonal, that is, all eigenvalues of J are distinct, and if they are complex, we have gone in to complex coordinates to put the eigenvalues on the diagonal. In this case, J is called semisimple – the non-semisimple case is discussed below.

Let λ_1, λ_2, ..., λ_n be the eigenvalues (diagonal entries) of J.

Our aim is to perform near-identity coordinate transformations to bring the nonlinear part of $f(x)$ into as simple a form as possible.

Note: often normal form calculations are done after the centre manifold reduction, so the equilibrium point $x = 0$ will be non-hyperbolic, but in fact normal form calculations can be done with any equilibrium point.

Birkhoff normal form IX

Third, we see what we can do to the terms in $f(x)$ that are of degree k. Write $f(x)$ as

$$\dot{x} = f(x) = Jx + f_k(x) + h_k(x) + \mathcal{O}(k+1)$$

where $f_k(x)$ contains terms of degree $2, \ldots, k-1$, $h_k(x)$ contains terms of degree k, and all higher-order terms are in $\mathcal{O}(k+1)$. We are aiming to make h_k as simple as possible; we assume that we've already done this for $f_k(x)$.

We carry out all the steps 3, 4, ... with $k = 2$ (quadratic terms) at first. With $k = 2$, we have $f_2 = 0$ and all the quadratic terms are in h_2. Once we have done what we can with these, we set $k = 3$, with $f_3 =$ remaining quadratic terms and $h_3 =$ cubic terms, and repeat steps 3, 4, ... Usually going to $k = 3$ is enough; if not, we repeat steps 3, 4, ... with $k = 4$, $k = 5$, ...

Birkhoff normal form X

Fourth, we perform a near-identity coordinate transformation of degree k: write

$$x = y + P_k(y),$$

where $P_k(y)$ is a homogeneous polynomial of degree k (all terms are of the same degree). Our task is to choose $P_k(y)$ to make the terms of degree k in the new ODE for y as simple as possible.

We need to invert the coordinate transformation:

$$y = x - P_k(y) = x - P_k(x - P_k(y)) = x - P_k(x - P_k(x - P_k(y))) = \dots$$

In the simplest case, we can write this as

$$y = x - P_k(x) + \mathcal{O}(k+1)$$

but we need to be aware that when we start (for example) with $k = 2$, this transformation will alter the cubic terms, and if the cubic terms turn out to be required in the final normal form, we need to go to cubic order in the inverse coordinate transformation.

Birkhoff normal form XI

Fifth, we differentiate this:

$$\dot{y} = \dot{x} - DP_k(x)\dot{x} + \mathcal{O}(k+1),$$

where $DP_k(x) = \partial P_k(x)/\partial x$ is the $n \times n$ matrix of partial derivatives; the entries in $DP_k(x)$ are polynomials of degree $k-1$.

Sixth, we insert the ODE for x:

$$\dot{y} = Jx + f_k(x) + h_k(x) - DP_k(x)Jx + \mathcal{O}(k+1),$$

where $DP_k(x)f_k(x)$ etc. are absorbed into $\mathcal{O}(k+1)$.

Seventh, we substitute $x = y + P_k(y)$ into the ODE:

$$\dot{y} = J(y + P_k(y)) + f_k(y) + h_k(y) - DP_k(y)Jy + \mathcal{O}(k+1),$$

where we have made use of the fact that

$$f_k(y+P_k(y)) = f_k(y) + \mathcal{O}(k+1), \qquad h_k(y+P_k(y)) = h_k(y) + \mathcal{O}(2k-1)$$

and $DP_k(y + P_k(y)) = DP_k(y) + \mathcal{O}(2k-2)$.

Birkhoff normal form XII

Eighth, we write this as:

$$\dot{y} = Jy + f_k(y) + (h_k(y) + JP_k(y) - DP_k(y)Jy) + \mathcal{O}(k+1),$$

and note that the terms $Jy + f_k(y)$ are unchanged from the original ODE for x, while the degree k term (h_k) can be altered by our choice of P_k.

The expression $L_J(P_k(y)) = DP_k(y)Jy - JP_k(y)$ is called the commutator of the vector fields $P_k(y)$ and Jy, or the Lie derivative operator associated with the matrix J:

$$L_J(P(y)) = DP(y)Jy - JP(y);$$

L_J is a linear map from homogeneous polynomial vector fields of degree k onto homogeneous polynomial vector fields of degree k.

If we can solve the equation $L_J(P_k(y)) = h_k(y)$, then we can eliminate terms of degree k from the ODE; if we can do this for all k, then we end up (formally) with the linear ODE $\dot{y} = Jy$.

Birkhoff normal form XIII

To see what L_J does, we define polynomials in terms of y^m: let $y = (y_1, y_2, \dots, y_n)$ with $y \in \mathbb{R}^n$ and let $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$. Define

$$y^m = y_1^{m_1} y_2^{m_2} \dots y_n^{m_n}.$$

We also let $|m| = m_1 + m_2 + \cdots + m_n$, so $|m|$ is the degree of the polynomial y^m. For example, if $n = 2$ and $m = (2, 1)$, with $|m| = 3$, then $y^m = y_1^2 y_2$, a cubic term. In fact, we are dealing with vectors, for example

$$P_k(y) = \left(P_k^{(1)}(y), P_k^{(2)}(y), \dots P_k^{(n)}(y)\right)^T$$

and similarly $h_k = \left(h^{(1)}, \dots, h^{(n)}\right)^T$. Each entry in the P_k vector is a polynomial of degree k (the $P_m^{(j)}$ are numbers):

$$P_k^{(j)} = \sum_{|m|=k} P_m^{(j)} y^m.$$

Birkhoff normal form XIV

Differentiating y^m with repect to y_j $(1 \leq j \leq n)$ yields

$$\frac{\partial y^m}{\partial y_j} = \frac{\partial y_1^{m_1} y_2^{m_2} \ldots y_n^{m_n}}{\partial y_j} = y_1^{m_1} y_2^{m_2} \ldots m_j y_j^{m_j - 1} \ldots y_n^{m_n} = m_j \frac{y^m}{y_j},$$

so we can compute the (i, j) entry in the matrix $DP_k(y)$ as

$$DP_k^{(i,j)}(y) = \frac{\partial P_k^{(i)}}{\partial y_j} = \sum_{|m|=k} m_j P_m^{(i)} \frac{y^m}{y_j}.$$

The ith entry in the first term in the Lie derivative operator is

$$(DP_k J y)^{(i)} = \sum_{|m|=k} \sum_{j=1}^{n} m_j \lambda_j P_m^{(i)} y^m$$

($\lambda_1, \ldots$ are the diagonal entries of J.) The ith entry in $JP_k(y)$ is

$$(JP_k)^{(i)} = \sum_{|m|=k} \lambda_i P_m^{(i)} y^m$$

Birkhoff normal form XV

Putting these together results in

$$\begin{aligned}\big(h_k(y) + JP_k(y) - DP_k(y) J y\big)^{(i)} \\ = \sum_{|m|=k} \left(h_m^{(i)} + \left(\lambda_i - \sum_{j=1}^{n} m_j \lambda_j \right) P_m^{(i)} \right) y^m\end{aligned}$$

Ninth, we recall that our aim was to choose the $P_m^{(i)}$ coefficients to make the new nonlinear term as simple as possible – ideally zero. If $\lambda_i \neq \sum_{j=1}^{n} m_j \lambda_j$, then we can choose $P_m^{(i)}$ appropriately and set the coefficient of y^m to zero.

However, if $\lambda_i = \sum_{j=1}^{n} m_j \lambda_j$ (there is a resonance between the eigenvalues), then the coefficient of y^m cannot be altered.

Birkhoff normal form XVI

Definition: The set of eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix J is called resonant if

$$\lambda_i = \sum_{j=1}^{n} m_j \lambda_j$$

for some set of non-negative integers $(m_1, \ldots, m_n)$; $|m|$ is called the order of the resonance.

For example:

- If $\lambda_1 = \lambda_2 + \lambda_3$, there is a resonance of order 2, and we will not be able to eliminate $y_2 y_3$ from the $\dot{y}_1$ equation.
- If $\lambda_1 = 0$, then there are resonances of order 2: $\lambda_i = \lambda_1 + \lambda_i$ (and all higher orders).
- If $\lambda_1 = -\lambda_2$, then there are resonances of order 3: $\lambda_i = \lambda_1 + \lambda_2 + \lambda_i$ (as well as all higher odd orders).

Birkhoff normal form XVII

Tenth, once we've eliminated as many nonlinear terms as possible by successive near-identity transformations of all orders, we are left with

$$\dot{y}^{(i)} = (Jy)^{(i)} + \sum_{k=2}^{\text{“}\infty\text{”}} \sum_{\substack{|m|=k \\ m \text{ resonant}}} h_m^{(i)} y^m,$$

where the $h_m^{(i)}$ are related to the coefficients in the original ODE $\dot{x} = f(x)$, and the sum is to “∞” indicates that this is a formal series – it may not converge. This is known as the Birkhoff normal form.

Usually the infinite sum is truncated at some order N – this is known as the truncated Birkhoff normal form. The behaviour of the truncated Birkhoff normal form is not guaranteed to be the same as that of the original ODE (but it is usually possible to identify the circumstances in which the behaviour will be different).

Birkhoff normal form XVIII

We've already done one example: the normal form calculation for the Hopf bifurcation at the start of this section. We ended up with:

$$\begin{aligned}\dot{u} &= v + (P_{21}+Q_{30})u^3 + (2P_{12}-3P_{30}+Q_{21})u^2v \\ &\quad + (3P_{03}-2P_{21}+Q_{12})uv^2 + (-P_{12}+Q_{03})v^3 + \mathcal{O}(4) \\ \dot{v} &= -u + (M-P_{30}+Q_{21})u^3 + (N-P_{21}+2Q_{12}-3Q_{30})u^2v \\ &\quad + (-P_{12}+3Q_{03}-2Q_{21})uv^2 + (-P_{03}-Q_{12})v^3) + \mathcal{O}(4),\end{aligned}$$

at the end of stage eight. Can we eliminate the cubic terms altogether? Answer: no, because of the resonance of the eigenvalues of J: $\pm i$ for this choice of parameters (exercise).

We'll do a second example: let $n=2$ and let the original ODE be

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}h^{(1)}_{20}x_1^2 + h^{(1)}_{11}x_1x_2 + h^{(1)}_{02}x_2^2\\ h^{(2)}_{20}x_1^2 + h^{(2)}_{11}x_1x_2 + h^{(2)}_{02}x_2^2\end{pmatrix},$$

Birkhoff normal form XIX

so $x=0$ is an equilibrium point, $J=\left[\begin{smallmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{smallmatrix}\right]$ (J is diagonal) and the nonlinear terms are $h_2(x)$. We take $k=2$ and write the near-identity transformation as

$$x = y + P_2(y) = y + \begin{pmatrix}P^{(1)}_{20}y_1^2 + P^{(1)}_{11}y_1y_2 + P^{(1)}_{02}y_2^2\\ P^{(2)}_{20}y_1^2 + P^{(2)}_{11}y_1y_2 + P^{(2)}_{02}y_2^2\end{pmatrix}.$$

We need to invert the coordinate transformation:

$$y = x - P_2(y) = x - P_2(x - P_2(y)) = x - P_2(x) + \mathcal{O}(3).$$

This accomplishes steps 1–4:

$$\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}x_1\\ x_2\end{pmatrix} - \begin{pmatrix}P^{(1)}_{20}x_1^2 + P^{(1)}_{11}x_1x_2 + P^{(1)}_{02}x_2^2\\ P^{(2)}_{20}x_1^2 + P^{(2)}_{11}x_1x_2 + P^{(2)}_{02}x_2^2\end{pmatrix} + \mathcal{O}(3).$$

Birkhoff normal form XX

Fifth, we differentiate this:

$$\begin{pmatrix}\dot y_1\\ \dot y_2\end{pmatrix} = \begin{pmatrix}\dot x_1\\ \dot x_2\end{pmatrix} - \begin{bmatrix} 2P_{20}^{(1)}x_1 + P_{11}^{(1)}x_2 & P_{11}^{(2)}x_1 + 2P_{02}^{(2)}x_2 \\ 2P_{20}^{(1)}x_1 + P_{11}^{(1)}x_2 & P_{11}^{(2)}x_1 + 2P_{02}^{(2)}x_2 \end{bmatrix} \begin{pmatrix}\dot x_1\\ \dot x_2\end{pmatrix} + \mathcal{O}(3),$$

where the term in [] is $DP_2(x)$, the 2×2 matrix of partial derivatives.

Sixth, seventh and eighth, we insert the ODE for x:

$$\dot y = Jx + h_2(x) - DP_2(x)Jx + \mathcal{O}(3)$$

and substitute $x = y + P_2(y)$ into the ODE:

$$\dot y = Jy + JP_2(y) + h_2(y) - DP_2(y)Jy + \mathcal{O}(3),$$

Birkhoff normal form XXI

or

$$\begin{pmatrix}\dot y_1\\ \dot y_2\end{pmatrix} = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} + \begin{pmatrix} y_1^2\left(h_{20}^{(1)} - P_{20}^{(1)}\lambda_1\right) + y_1y_2\left(h_{11}^{(1)} - P_{11}^{(1)}\lambda_2\right) + y_2^2\left(h_{02}^{(1)} + P_{02}^{(1)}(\lambda_1 - 2\lambda_2)\right) \\ y_1^2\left(h_{20}^{(2)} + P_{20}^{(2)}(\lambda_2 - 2\lambda_1)\right) + y_1y_2\left(h_{11}^{(2)} - P_{11}^{(2)}\lambda_1\right) + y_2^2\left(h_{02}^{(2)} - P_{02}^{(2)}\lambda_2\right)\end{pmatrix} + \mathcal{O}(3).$$

Ninth, which nonlinear terms can we eliminate by our choice of the P coefficients? It depends on whether there are any resonances of order two (that is, whether any of the coefficients of the P's are zero).

Birkhoff normal form XXII

- If $\lambda_1 = 0$ but $\lambda_2 \neq 0$ ($\lambda_1 = \lambda_1 + \lambda_1$, $\lambda_2 = \lambda_1 + \lambda_2$), we get:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h_{20}^{(1)} y_1^2 \\ h_{11}^{(2)} y_1 y_2 \end{pmatrix} + \mathcal{O}(3),$$

  with a similar result if $\lambda_2 = 0$ and $\lambda_1 \neq 0$.
- If $\lambda_1 = 2\lambda_2 \neq 0$ ($\lambda_1 = \lambda_2 + \lambda_2$), we get:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{bmatrix} 2\lambda_2 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h_{02}^{(1)} y_2^2 \\ 0 \end{pmatrix} + \mathcal{O}(3),$$

  with a similar result if $\lambda_2 = 2\lambda_1 \neq 0$.
- If $\lambda_1 = \lambda_2 = 0$, we can't eliminate any of the quadratic terms.
- If none of the conditions above hold, we can eliminate all of the quadratic terms.

Birkhoff normal form XXIII

Tenth, we have eliminated some or all quadratic terms and ended up with a simpler ODE that will in principle be easier to analyse than the original ODE, but the cost is that we have introduced cubic terms. Going to higher order will eliminate some of these. Typically we would truncate at some order, resulting in the truncated Birkhoff normal form.

The general discussion has all been for the semisimple case, where the Jacobian matrix does not have any repeated eigenvalues. We won't cover the non-semisimple case in detail, but there is an example on the exercise sheet 2. We will note that in this case there isn't an obvious choice to make, and the normal form is not unique. In the example on the exercise sheet, we start with six quadratic nonlinear terms and

- we can set four of these to zero; or

Birkhoff normal form XXIV

- we can choose the coordinate transformation such that the nonlinear part of the normal form commutes with $\exp\left(sJ^T\right)$ for all s. This results in setting three coefficients to zero.

"the nonlinear part of the normal form commutes with $\exp\left(sJ^T\right)$" means that if

$$\dot{y} = Jy + g_k(y) + \mathcal{O}(k+1)$$

after the coordinate transformation, where $g_k(y)$ contains nonlinear terms from degree 2 up to and including k, then

$$g_k\left(\exp\left(sJ^T\right)y\right) = \exp\left(sJ^T\right)g_k(y),$$

for all s and for all y, where $\exp\left(sJ^T\right)$ is defined in terms of the Taylor series of the exponential:

$$\exp\left(sJ^T\right) = I + sJ^T + \frac{1}{2}\left(sJ^T\right)^2 + \dots$$

Birkhoff normal form XXV

If in addition, the linear part of the normal form commutes with $\exp\left(sJ^T\right)$, then $\exp\left(sJ^T\right)$ is said to be a normal form symmetry. The Hopf normal form is an example: in this case,

$$J = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

and

$$\exp(sJ^T) = \begin{bmatrix} \cos(s\omega) & \sin(s\omega) \\ -\sin(s\omega) & \cos(s\omega) \end{bmatrix},$$

which is a rotation. Rotations commute with J, so the normal form will have rotation symmetry, and so can only contain terms like

$$(y_1^2 + y_2^2)^k \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

which reduce to $\dot{r} = \dots + Ar^{2k+1} + \dots$ and $\dot{\theta} = \dots + Br^{2k} + \dots$

Birkhoff normal form XXVI

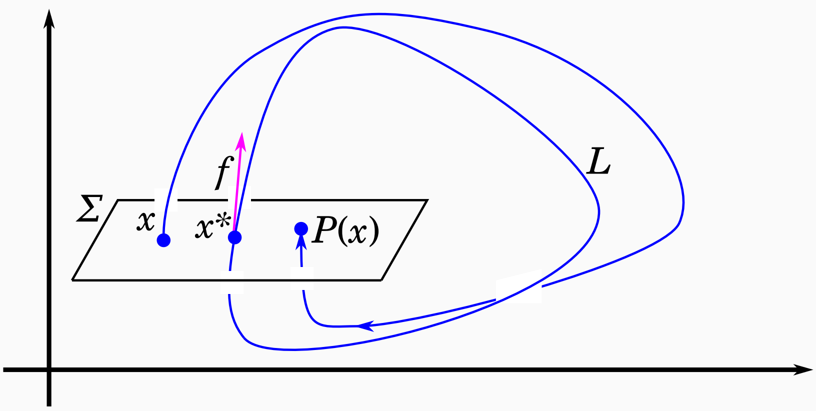
Some final remarks:

- In general, we could include parameters in the normal form calculations, but we need to be aware that resonances can come and go as the eigenvalues change. In particular, having a zero or pure imaginary eigenvalue will introduce low-order resonances.
- A better way is to work out the normal form at the bifurcation point and then re-introduce parameters by unfolding. See Guckenheimer & Holmes (1983) for example.

Local bifurcations of periodic orbits I

Much of the analysis that we've done for equilibrium points (structural stability, stable and unstable manifolds, centre manifold reduction, local bifurcation theory, Bikhoff normal forms) carries over to periodic orbits, with some similarities and some differences.

Recall: L is a periodic orbit with period T_L, and x^* is a point on L. Take a cross-section $\Sigma = \{x \in X | (x - x^*) \cdot f(x^*) = 0\}$, and define the Poincaré map as before: $x \to P(x) = \Phi^{T(x)}(x)$, where $T(x)$ is the time it takes for the orbit of x to return to Σ, close to x^*.



Local bifurcations of periodic orbits II

Instead of linearising the Poincaré map, we redefine our coordinates so that the fixed point x^* of the map is at the origin:

$$x \to P(x;\mu) = Mx + \mathcal{O}(x^2).$$

We also rotate our coordinate system so that the monodromy matrix M (the linearised Poincaré map) is in Jordan normal form.

Recall that the eigenvalues of the monodromy matrix are called the Floquet multipliers of the periodic orbit, and these determine its stability: if any are greater than one in magnitude, perturbations will grow, and if they are all less than one in magnitude (apart from the one Floquet multiplier that is forced to be equal to one), perturbations will decay.

In general, the Floquet multipliers will depend on the parameter μ.

Local bifurcations of periodic orbits III

The periodic orbit is non-hyperbolic when:

- A Floquet multiplier equals 1;
- A Floquet multiplier equals -1;
- A pair of Floquet multipliers equal $e^{\pm i\alpha_0}$.

In each case, we can (in principle) perform an extended centre manifold reduction and bring the dimension of the map down to one (first two cases) or two (last case).

Floquet multiplier equal to 1: in this case, the periodic orbit undergoes a saddle-node bifurcation, described by the normal form:

$$x \to P(x;\mu) = \mu + x - x^2$$

(as well as variants with other choices of sign). The fixed points satisfy

$$x = P(x;\mu), \qquad \text{so } x = \pm\sqrt{\mu}.$$

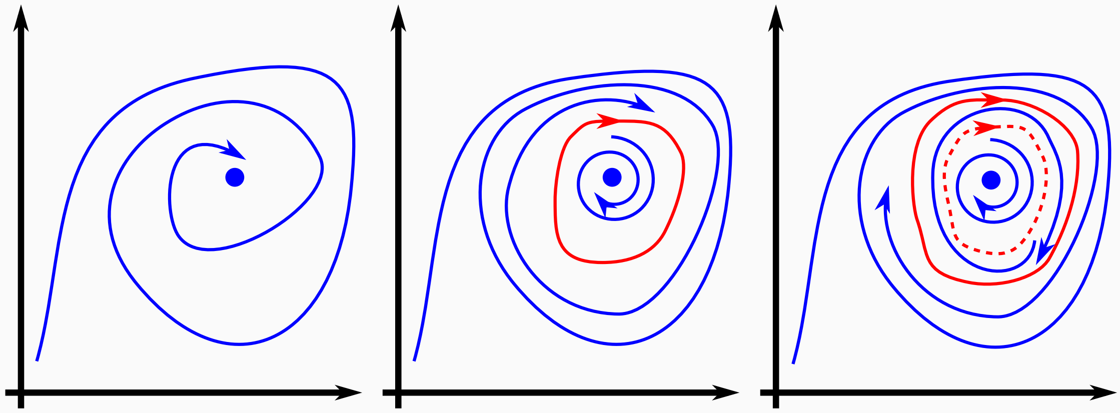
Local bifurcations of periodic orbits IV

The monodromy matrix is

$$M = \left[\frac{\partial P}{\partial x}\right] = \left[1 - 2x\Big|_{x=\pm\sqrt{\mu}}\right] = \left[1 \mp 2\sqrt{\mu}\right],$$

so there is a Floquet multiplier (eigenvalue of M) equal to 1 when $\mu = 0$, and the fixed point $x = +\sqrt{\mu}$) is stable for small positive μ:

$\mu < 0$ $\mu = 0$ $\mu > 0$



Transcritical and pitchfork bifurcations are also possible.

Local bifurcations of periodic orbits V

Floquet multiplier equal to -1: in this case, the periodic orbit undergoes a period-doubling (or flip) bifurcation, described by the normal form:

$$x \to P(x;\mu) = -(1+\mu)x + x^3$$

(as well as variants with other choices of sign). Note that this has a normal form symmetry: $P(-x;\mu) = -P(x;\mu)$. The fixed points satisfy

$$x = P(x;\mu), \qquad \text{so } x = 0.$$

The monodromy matrix is

$$M = \left[\frac{\partial P}{\partial x}\right] = \left[-(1+\mu) + 3x^2\Big|_{x=0}\right] = \left[-1-\mu\right],$$

so there is a Floquet multiplier (eigenvalue of M) equal to -1 when $\mu = 0$, and the fixed point $x = 0$ is stable for small negative μ.

Local bifurcations of periodic orbits VI

There are period-2 points that satisfy $x = P(P(x;\mu);\mu)$; this amounts to a ninth-degree polynomial equation to solve. If x^* is a solution, trajectories would go $x^* \to P(x^*;\mu) \to x^* \to \dots$.
We can use the odd normal form symmetry of P to find symmetric period-two points, which go $x^* \to -x^* \to x^* \to \dots$ instead, and so satisfy

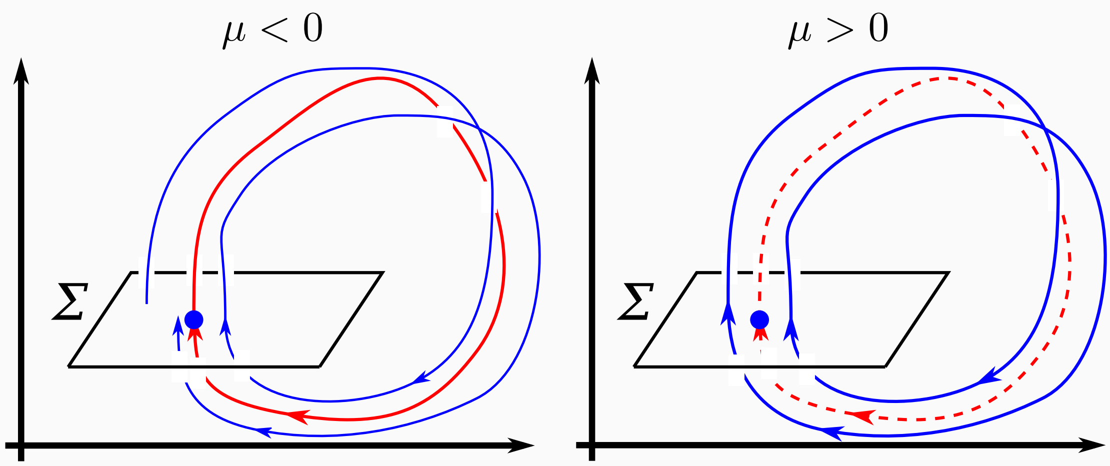
$$x = -P(x;\mu) = (1+\mu)x - x^3, \qquad \text{so } x = \pm\sqrt{\mu}.$$

These are period-two points because $x^* = P(P(x^*;\mu);\mu)$ (making use of the odd normal form symmetry of P). Their stability is given by the monodromy matrix of the period-doubled map:

$$\left[\frac{\partial P(P(x))}{\partial x}\right] = \left[\left.\frac{\partial P(x)}{\partial x}\right|_{x=\sqrt{\mu}}\right]\left[\left.\frac{\partial P(x)}{\partial x}\right|_{x=-\sqrt{\mu}}\right] = \left[(1-2\mu)^2\right],$$

so the period-two points are stable for small positive μ.

Local bifurcations of periodic orbits VII



- Unlike the pitchfork bifurcation, the period-doubling bifurcation is structurally stable.
- Period-doubling bifurcations in flows need $n \geq 3$ dimensions, so a centre-manifold reduction is necessary (in principle) to reduce the $n-1$ dimensional Poincaré map to the one-dimensional period-doubling normal form.
- The gap between the two blue loops is a Möbius strip.

Local bifurcations of periodic orbits VIII

Floquet multipliers equal to $e^{\pm i\alpha_0}$: in this case, the periodic orbit undergoes a torus (or Neimark–Sacker) bifurcation, described by the normal form in complex coordinates:

$$z \to P(z;\mu) = (1+\mu)e^{i\alpha}z + (A+iB)e^{i\alpha}|z|^2 z,$$

where α, A and B depend on μ, with $\alpha(0) = \alpha_0$. Note that this has a normal form symmetry: $P(e^{i\phi}z) = e^{i\phi}P(z)\,\forall\phi$.

The fixed point $z = 0$ has Floquet multipliers $(1+\mu)e^{i\alpha(\mu)}$, so at $\mu = 0$, the Floquet multipliers are $e^{i\alpha_0}$ (and its complex conjugate): the fixed point is non-hyperbolic.

In normal form, we can separate the dynamics of the modulus and phase of z. Write $z = re^{i\theta}$ and we obtain

$$\begin{aligned} r &\to r\left|(1+\mu) + (A+iB)r^2\right|, \\ \theta &\to \theta + \alpha(\mu) + \arg\left((1+\mu) + (A+iB)r^2\right) \end{aligned}$$

or

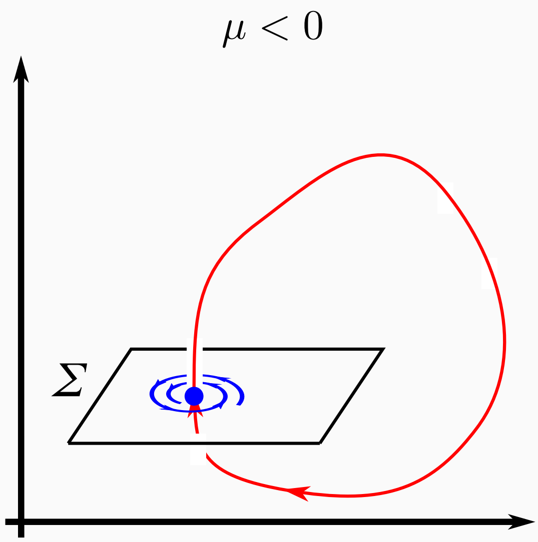
Local bifurcations of periodic orbits IX

$$\begin{aligned} r &\to (1+\mu)r + Ar^3 + \mathcal{O}(r^4), \\ \theta &\to \theta + \alpha(\mu) + \mathcal{O}(r^2) \end{aligned}$$

Note that r has decoupled from θ, and we find fixed points of the r map at $r = 0$ and $r = \sqrt{-\mu/A}$ – this represents a closed invariant curve, supercritical if $A < 0$.

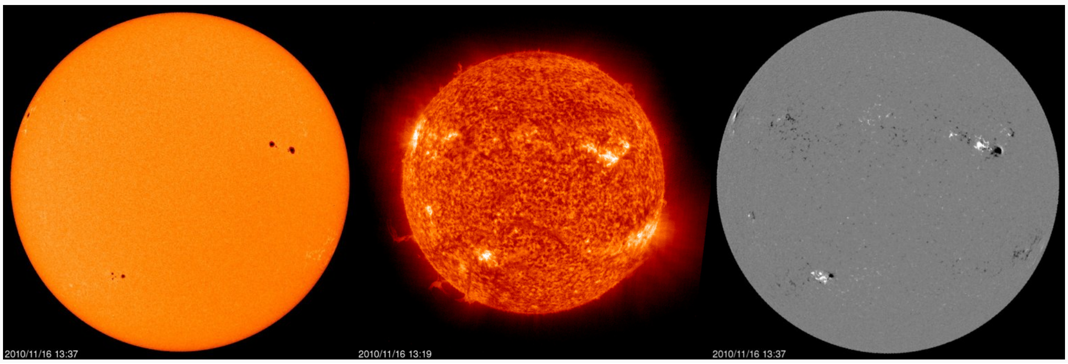
The θ dynamics is an r-dependent circle map: orbits on the invariant circle are not periodic if $\alpha(\mu) + \mathcal{O}(r^2)$ is an irrational multiple of 2π; conversely, if $\alpha(\mu) + \mathcal{O}(r^2)$ is $\frac{p}{q}2\pi$, then all points on the invariant curve are periodic with period q. This situation is structurally unstable (there are infinitely many resonances), and the resolution is beyond the scope of this course (see MAGIC060: Dynamical Systems II (maps) perhaps).

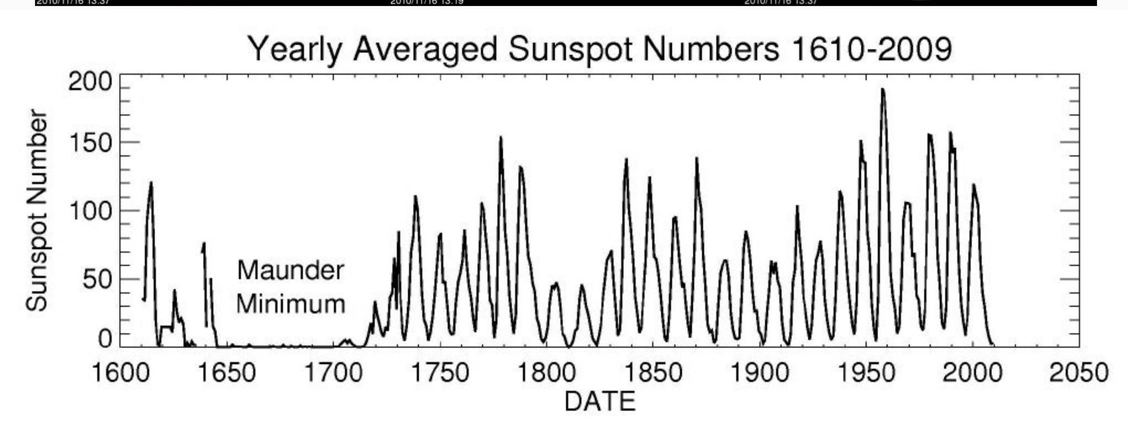
Local bifurcations of periodic orbits X



- On the Poincaré section, trajectories change from spiralling in to going in circles, while in the flow, trajectories change from spiralling in to the red periodic orbit to following in a torus around that periodic orbit.
- On the torus, trajectories may be periodic or quasiperiodic.

Example: saddle-node–Hopf bifurcation I





http://solarscience.msfc.nasa.gov/

Example: saddle-node–Hopf bifurcation II

The model (Tobias *et al.* (1995), Wilmot-Smith *et al.* (2005)): the Sun has two hydrodynamic states, one that is a dynamo (it is unstable to the generation of magnetic field) and one that is not. The dynamo instability is a Hopf bifurcation whose frequency corresponds to the 22-year solar magnetic cycle. The magnetic cycle does not saturate to a constant amplitude periodic orbit; instead, its amplitude is modulated on a long time scale, with the possibility of being switched off for a time.

At its simplest, the model is a coupling of a two normal forms. The first is for a saddle-node bifurcation:

$$\dot{x} = \mu - x^2 + \dots$$

so $x = \pm\sqrt{\mu}$ represent the two hydrodynamic states of the Sun.

Example: saddle-node–Hopf bifurcation III

The second is a Hopf bifurcation for the magnetic field:

$$\dot{z} = (\lambda + i\omega)z + \dots,$$

where the real and imaginary parts of z represent two components of the magnetic field of the Sun. We'd like λ to depend on x in such a way that one hydrodynamic state is a dynamo (z grows), the other is not (z decays).

What nonlinear terms would be appropriate for this model? One approach is to think in terms of a codimension-two saddle-node–Hopf bifurcation: we set $\mu = \lambda = 0$ and compute the Birkhoff normal form, based on the linearisation:

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i\omega & 0 \\ 0 & 0 & -i\omega \end{bmatrix}.$$

Example: saddle-node–Hopf bifurcation IV

The three eigenvalues of J are 0 and $\pm i\omega$, so we expect resonances of order 2 (from the zero eigenvalue) and of order 3 (from the $\pm i\omega$ eigenvalues).

Thus the normal form will be (up to cubic order)

$$\begin{aligned}\dot{x} &= \mu + h^{(1)}_{200}x^2 + h^{(1)}_{011}|z|^2 + h^{(1)}_{300}x^3 + h^{(1)}_{111}x|z|^2 + \dots \\ \dot{z} &= (\lambda + i\omega)z + h^{(2)}_{110}xz + h^{(2)}_{210}x^2z + h^{(2)}_{021}|z|^2z + \dots,\end{aligned}$$

where $h^{(1)}_{ijk}$ are real and $h^{(2)}_{ijk}$ are complex functions of the parameters μ and λ. However, additional transformations (see Kuznetsov 8.5) allow the removal of three of the four cubic terms – which ones are removed are a matter of taste. Here, we will keep $h^{(1)}_{210}x^2z$.

Throughout, we will use a phase plane plotter (for example, `http://math.rice.edu/~dfield/dfpp.html`) to explore further.

Example: saddle-node–Hopf bifurcation V

In addition, we write $h^{(2)}_{110} = A + iB$, and we scale x, z and t so that $h^{(1)}_{200} = -1$ (to keep the usual saddle-node normal form), $h^{(1)}_{011} = -1$ (the magnetic force, modelled by $-|z|^2$, acts to suppress fluid motion), and $h^{(2)}_{210} = -1$. This results in (truncating at cubic order):

$$\begin{aligned}\dot{x} &= \mu - x^2 - |z|^2 \\ \dot{z} &= (\lambda + i\omega)z + (A + iB)xz - x^2z\end{aligned}$$

Note that we have the normal form symmetry that allows us to go into polar coordinates. Writing $z = re^{i\theta}$, we get:

$$\begin{aligned}\dot{x} &= \mu - x^2 - r^2 \\ \dot{r} &= r(\lambda + Ax - x^2) \\ \dot{\theta} &= \omega + Bxr.\end{aligned}$$

We drop the last equation since the first two do not depend on θ.

Example: saddle-node–Hopf bifurcation VI

We proceed first by dropping the one cubic term (we will restore it later) so the system is:

$$\dot{x} = \mu - x^2 - r^2$$
$$\dot{r} = r(\lambda + Ax).$$

We seek equilibrium points and their stability. One pair are $(x, r) = (\pm\sqrt{\mu}, 0)$ (these are the pure hydrodynamic states, and they are created in a saddle-node bifurcation at $\mu = 0$). The Jacobian matrices are

$$J(\sqrt{\mu}, 0) = \begin{bmatrix} -2\sqrt{\mu} & 0 \\ 0 & \lambda + A\sqrt{\mu} \end{bmatrix}, \quad J(-\sqrt{\mu}, 0) = \begin{bmatrix} 2\sqrt{\mu} & 0 \\ 0 & \lambda - A\sqrt{\mu} \end{bmatrix}$$

so for small μ and λ, and for $A > 0$, the first equilibrium is hydrodynamically stable ($-2\sqrt{\mu} < 0$) and can act as a dynamo if $\lambda + A\sqrt{\mu} > 0$, while the second is hydrodynamically unstable ($2\sqrt{\mu} > 0$) and does not act as a dynamo if $\lambda - A\sqrt{\mu} < 0$.

Example: saddle-node–Hopf bifurcation VII

There are additional equilibria with $r \neq 0$: these correspond to periodic orbits in the original system. They satisfy:

$$\lambda + Ax = 0 \qquad \text{or} \qquad x = -\frac{\lambda}{A}$$

and

$$r^2 = \mu - x^2 = \mu - \frac{\lambda^2}{A^2}$$

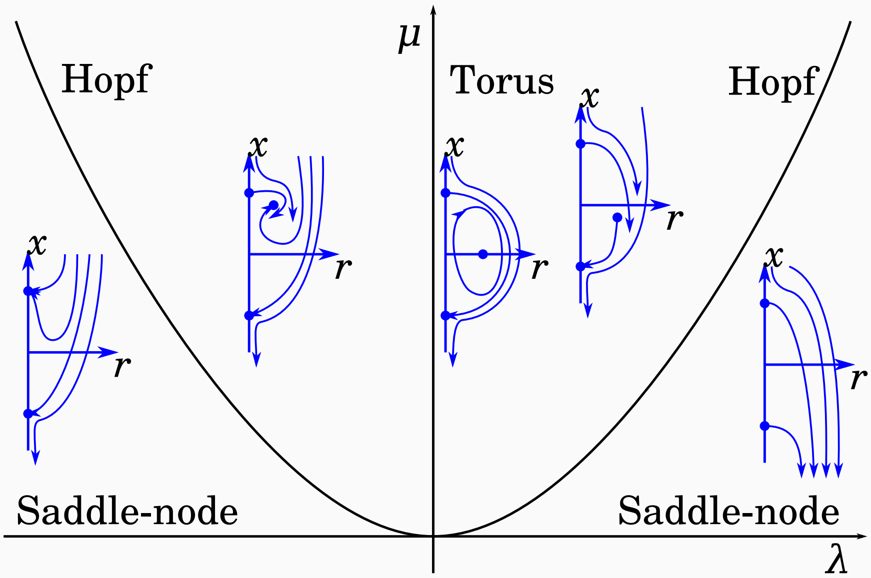
so they exist for $\mu > \lambda^2/A^2$ – this is a line of Hopf bifurcations in the original system, and it is tangent to the line of saddle-node bifurcations ($\mu = 0$) at $(\mu, \lambda) = (0, 0)$. Stability is given by

$$J\left(-\frac{\lambda}{A}, \sqrt{\mu - \frac{\lambda^2}{A^2}}\right) = \begin{bmatrix} 2\frac{\lambda}{A} & -2\sqrt{\mu - \frac{\lambda^2}{A^2}} \\ A\sqrt{\mu - \frac{\lambda^2}{A^2}} & 0 \end{bmatrix}.$$

Example: saddle-node–Hopf bifurcation VIII

$$\text{with} \qquad \text{Tr}(J) = 2\frac{\lambda}{A}, \qquad \text{Det}(J) = 2A\left(\mu - \frac{\lambda^2}{A^2}\right)$$

so there is a Hopf (torus) bifurcation from this periodic orbit when $\lambda = 0$ as long as $\mu A > 0$:



Example: saddle-node–Hopf bifurcation IX

Without the cubic term, the Hopf (torus) bifurcation is degenerate: at $\lambda = 0$, there is a nested family of periodic orbits. This is an indication that the system is integrable: if we define

$$E(x, r) = \frac{A}{2} r^{2/A} \left(\mu - x^2 - \frac{1}{A+1} r^2\right)$$

then

$$\begin{aligned}\frac{dE}{dt} &= \frac{\partial E}{\partial x}\dot{x} + \frac{\partial E}{\partial r}\dot{r} \\ &= \left(-r^{2/A} A x\right)(\mu - x^2 - r^2) + \left(r^{2/A-1}(\mu - x^2 - r^2)\right)(Arx) \\ &= 0.\end{aligned}$$

Thus when $\lambda = 0$, $E(x, r)$ is a constant of the motion. This situation is structurally unstable: the Hopf bifurcation coincides with the creation of a heteroclinic orbit, and any generic perturbation of the system will destroy this integrability.

Example: saddle-node–Hopf bifurcation X

The curve $E = 0$ corresponds to the heteroclinic connection between $(x, r) = (\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$: when $E = 0$, we have $r = 0$ (the x-axis) or

$$\mu - x^2 - \frac{1}{A+1} r^2 = 0,$$

an ellipse that goes through $(x, r) = (\pm\sqrt{\mu}, 0)$. In the original system, $E = 0$ is an ellipsoid.

Restoring the cubic term will destroy the integrability, but we can use the near-integrability to work out the location of the heteroclinic bifurcation (the Melnikov method). First, we restore the cubic term:

$$\begin{aligned} \dot{x} &= \mu - x^2 - r^2 \\ \dot{r} &= r(\lambda + Ax - x^2) \end{aligned}$$

Example: saddle-node–Hopf bifurcation XI

This system has

$$\begin{aligned} \frac{dE}{dt} &= \frac{\partial E}{\partial x}\dot{x} + \frac{\partial E}{\partial r}\dot{r} \\ &= \left(-r^{2/A} A x\right)\left(\mu - x^2 - r^2\right) + r^{2/A-1}(\mu - x^2 - r^2) r \left(\lambda + Ar - x^2\right) \\ &= r^{2/A}(\mu - x^2 - r^2)\left(\lambda - x^2\right). \end{aligned}$$

We let Γ_0 be the heteroclinic orbit connecting $(x, r) = (\pm\sqrt{\mu}, 0)$ with $\lambda = 0$ and no cubic term; in principle, we have $(x(t), r(t))$ on Γ_0. At $t = -\infty$, we have $(x, r) = (\sqrt{\mu}, 0)$ and $E = 0$, and at $t = +\infty$, we have $(x, r) = (-\sqrt{\mu}, 0)$ and $E = 0$. It follows that

$$0 = \int_{-\infty}^{\infty} \frac{dE}{dt}\, dt = \int_{-\infty}^{\infty} r^{2/A}\left(\mu - x^2 - r^2\right)\left(\lambda - x^2\right)\, dt$$

Since we know (in principle) $x(t)$ and $r(t)$ we can evaluate this integral and obtain a relation between μ and λ for the heteroclinic connection.

Example: saddle-node–Hopf bifurcation XII

However, for general A, this is intractable. Choosing $A = 2$ allows progress to be made. In this case, the curve $E = 0$ corresponds to the ellipse

$$\mu = x^2 + \frac{1}{3}r^2,$$

so $r = 3\sqrt{\mu - x^2}$, and (from the equation for $\dot{x}$)

$$dx = (\mu - x^2 - r^2)\, dt,$$

so the integral simplifies to

$$0 = \int_{-\sqrt{\mu}}^{\sqrt{\mu}} 3\sqrt{\mu - x^2}\left(\lambda - x^2\right)\, dx.$$

Example: saddle-node–Hopf bifurcation XIII

This integral can be evaluated by substituting $x = \sqrt{\mu}\cos\theta$, so $dx = -\sqrt{\mu}\sin\theta$, and the integral runs from $\theta = \pi$ to $\theta = 0$:

$$0 = \int_0^{\pi} 3\sqrt{\mu}\sin\theta\left(\lambda - \mu\cos^2\theta\right)\sqrt{\mu}\sin\theta\, d\theta.$$

or

$$0 = 3\mu\int_0^{\pi}\left(\lambda\sin^2\theta - \mu\sin^2\theta\cos^2\theta\right)\, d\theta = \frac{3\mu\pi}{8}(\lambda - 4\mu),$$

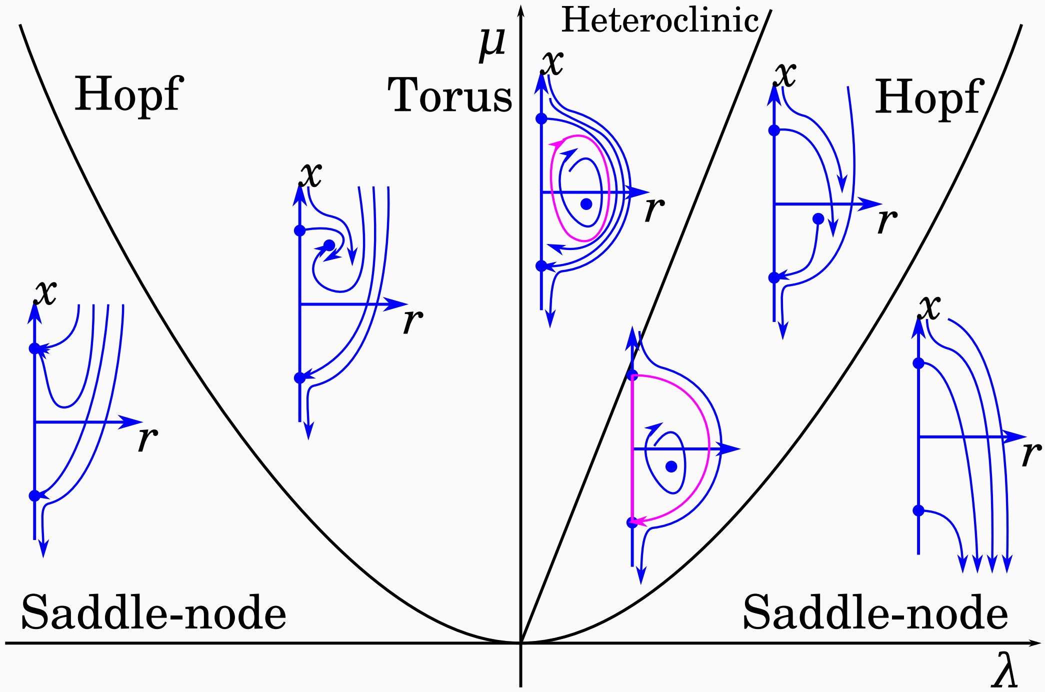
so the heteroclinic bifurcation occurs at

$$\lambda = 4\mu + \mathcal{O}(\mu^2).$$

A similar result holds even for $A \neq 2$.

Example: saddle-node–Hopf bifurcation XIV

The revised bifurcation diagram is:



Example: saddle-node–Hopf bifurcation XV

We are now in a position to interpret this as a model of the solar dynamo. First, in the original three-dimensional phase space, the torus solutions look like:

Copyrighted figure omitted

Tobias, Weiss & Kirk, *Chaotically modulated stellar dynamos* Mon. Not. R. Astron. Soc. **273** 1150–1166 (1995)

As a function of time, the "magnetic activity" in the model is:

Copyrighted figure omitted

Tobias, Weiss & Kirk (1995)

Compare with the real data:

Copyrighted figure omitted

The model is only capable of periodic or quasiperiodic time dependence, because of the reduction from three dimensions to two – this arises because of the normal form symmetry.

Copyrighted figure omitted

Kuznetsov (1998)

Example: saddle-node–Hopf bifurcation XVI

However, the system with this normal form symmetry is structurally unstable, since at the heteroclinic bifurcation, the two-dimensional unstable manifold of $(\sqrt{\mu}, 0)$ coincides exactly with the two-dimensional stable manifold of $(-\sqrt{\mu}, 0)$.

Copyrighted figure omitted

Kuznetsov (1998)

This is an extremely degenerate structure that disappears once higher-order terms are added, leading to heteroclinic tangencies and chaotic dynamics.

Copyrighted figure omitted

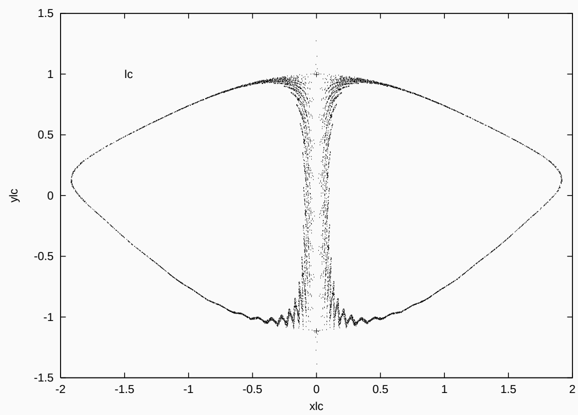
Kuznetsov (1998)

Example: saddle-node–Hopf bifurcation XVII

For example, with a z^3 term added to the $\dot{z}$ equation (this term retains the $z \to -z$ symmetry of magnetic fields), we get:

$$\dot{x} = \mu - x^2 - |z|^2$$
$$\dot{z} = (\lambda + i\omega)z + (A + iB)xz - x^2 z + Cz^3$$

Solutions can look like:



Ashwin, Rucklidge & Sturman, Physica **194D** 30–48 (2004)

Example: saddle-node–Hopf bifurcation XVIII

Copyrighted figure omitted

Tobias, Weiss & Kirk (1995)

Copyrighted figure omitted

The real system is rather more chaotic than this simple third-order set of ODEs – but the real system is a $1\,400\,000$ km diameter ball of boiling plasma!

Global bifurcations I

Recall that dynamical systems are structurally unstable whenever:

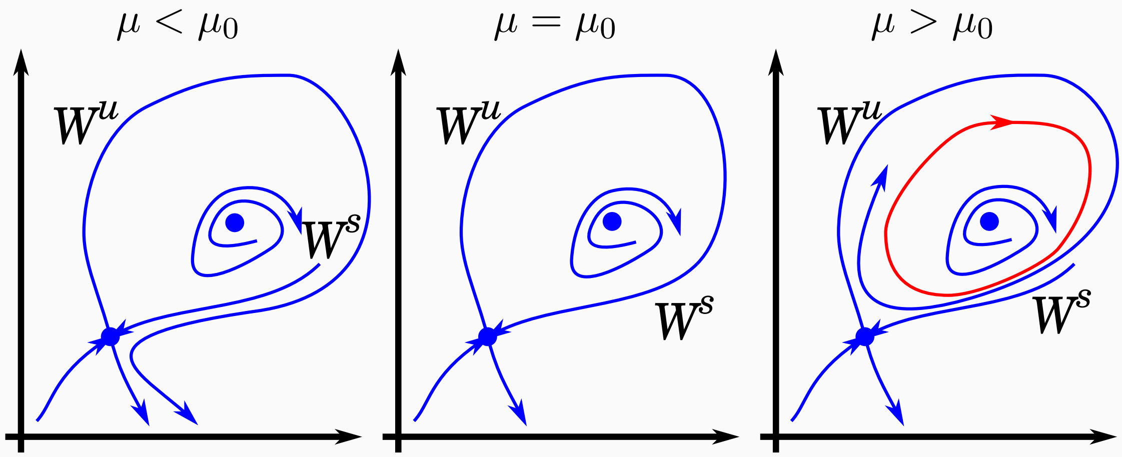
- an equilibrium point (or a periodic orbit) is non-hyperbolic: an eigenvalue with zero real part (or a Floquet multiplier on the unit circle) leads to a local bifurcation;
- the stable and unstable manifolds of an equilibrium point (or a periodic orbit) intersect non-transversally, leading to a global bifurcation.

We met an example of a global bifurcation (a heteroclinic bifurcation) in the saddle-node–Hopf normal form. Global bifurcations are often associated with the creation (or destruction) of periodic orbits and chaotic dynamics.

Global bifurcations come in two flavours: homoclinic bifurcations and heteroclinic bifurcations.

Global bifurcations II

Example: a global bifurcation, where the stable and unstable manifolds of an equilibrium are rearranged at $\mu = \mu_0$, and a periodic orbit is created.



At $\mu = \mu_0$, the stable/unstable manifolds of the equilibrium point make a homoclinic orbit.

Global bifurcations III

Definitions: the trajectory $\Gamma(x_0)$ (forwards and backwards) of a point x_0 is called homoclinic to an equilibrium point x^* if

$$\lim_{t\to\pm\infty} \Phi^t(x_0) = x^*.$$

$\Gamma(x_0)$ is called a heteroclinic connection between x_1^* and x_2^* if

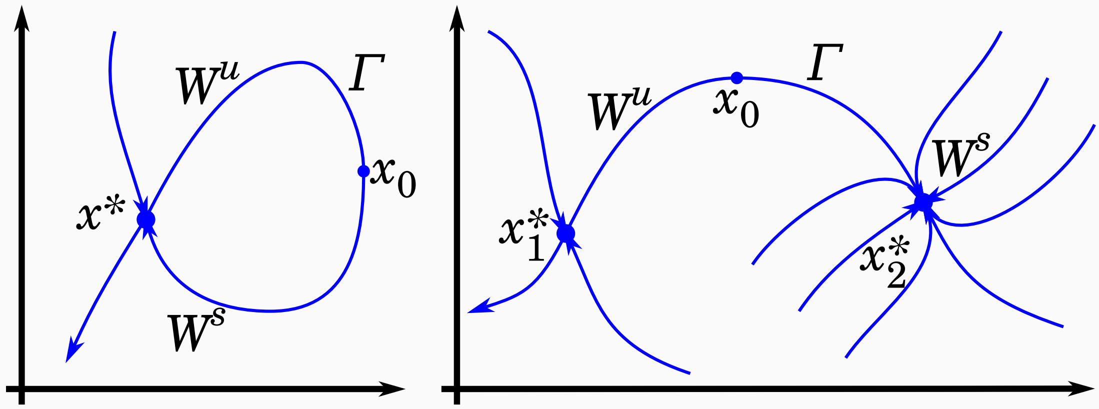
$$\lim_{t\to-\infty} \Phi^t(x_0) = x_1^* \quad \text{and} \quad \lim_{t\to+\infty} \Phi^t(x_0) = x_2^*.$$

The orbit $\Gamma(x_0)$ lies in the intersection of $W^u(x^*)$ and $W^s(x^*)$ (homoclinic), or of $W^u(x_1^*)$ and $W^s(x_2^*)$ (heteroclinic).

Recall that two manifolds M_1 and M_2 of dimensions m_1 and m_2 in $\mathbb{R}^n$ can intersect transversally if $m_1 + m_2 > n$ (think of two spheres in $\mathbb{R}^3$). Transversal intersections do not disappear if the system is perturbed slightly. Homoclinic orbits are always structurally unstable if x^* is hyperbolic.

Global bifurcations IV

Examples in $n = 2$ dimensions:

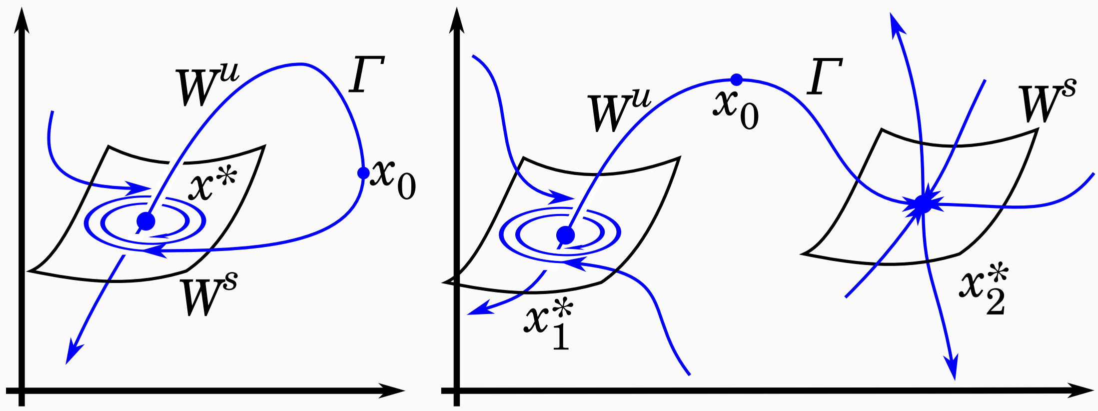


Left: homoclinic orbit from x^*, with $\dim(W_u) = \dim(W^s) = 1$, $\dim(W_u) + \dim(W^s) = n$ (non-transversal).

Right: heteroclinic orbit from x_1^* (a saddle) to x_2^* (a sink), with $\dim(W_u(x_1^*)) = 1$, $\dim(W^s(x_2^*)) = 2$, $\dim(W_u) + \dim(W^s) > n$ (transversal). If x_2^* had been a saddle, the intersection would have been non-transversal.

Global bifurcations V

Examples in $n = 3$ dimensions:



Left: homoclinic orbit from x^*, with $\dim(W_u) = 1$, $\dim(W^s) = 2$, $\dim(W_u) + \dim(W^s) = n$ (non-transversal).

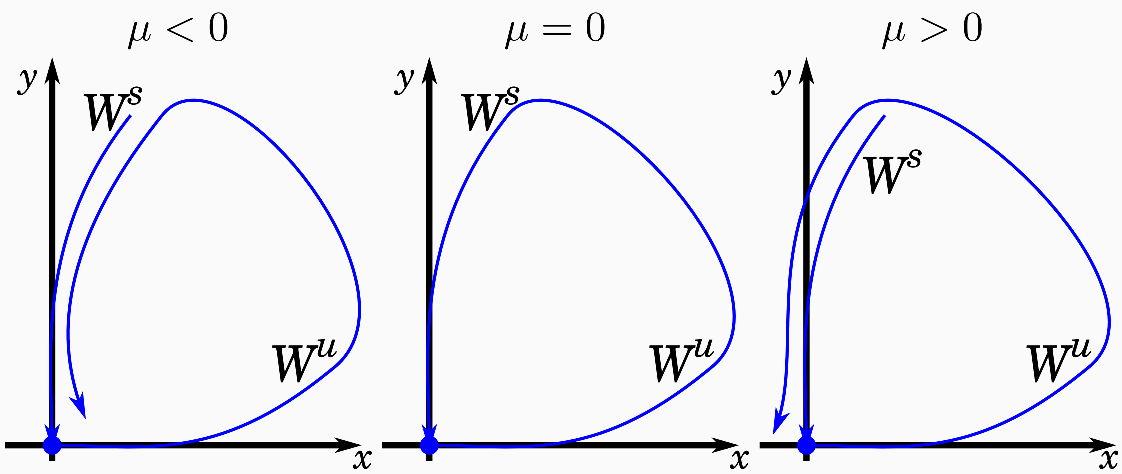
Right: heteroclinic orbit from x_1^* (a saddle) to x_2^* (a saddle), with $\dim(W_u(x_1^*)) = 1$, $\dim(W^s(x_2^*)) = 2$, $\dim(W_u) + \dim(W^s) = n$ (non-transversal). If x_2^* had been a sink, the intersection would have been transversal.

Global bifurcations VI

We will treat only homoclinic bifurcations in 2 dimensions in detail.

- Choose the coordinates so that the eqm point x^* is at $(0,0)$.
- Since there is a homoclinic orbit from x^* to itself, the equilibrium point is necessarily a saddle, and therefore has two real eigenvalues of opposite sign.
- We choose coordinates so that the linear unstable manifold E^u is the x-axis and the linear stable manifold E^s is the y-axis.
- There is a parameter μ, and at $\mu = 0$ there is a homoclinic orbit connecting the branch of W^u emerging from the positive x-axis to the branch of W^s going in to the positive y-axis.
- We choose μ such that for $\mu < 0$, the unstable manifold returns to a neighbourhood of x^* with a positive value of x, and for $\mu > 0$, it returns with a negative value of x.
- We will work through the global bifurcation theorem of Andronov and Leontovich (1939).

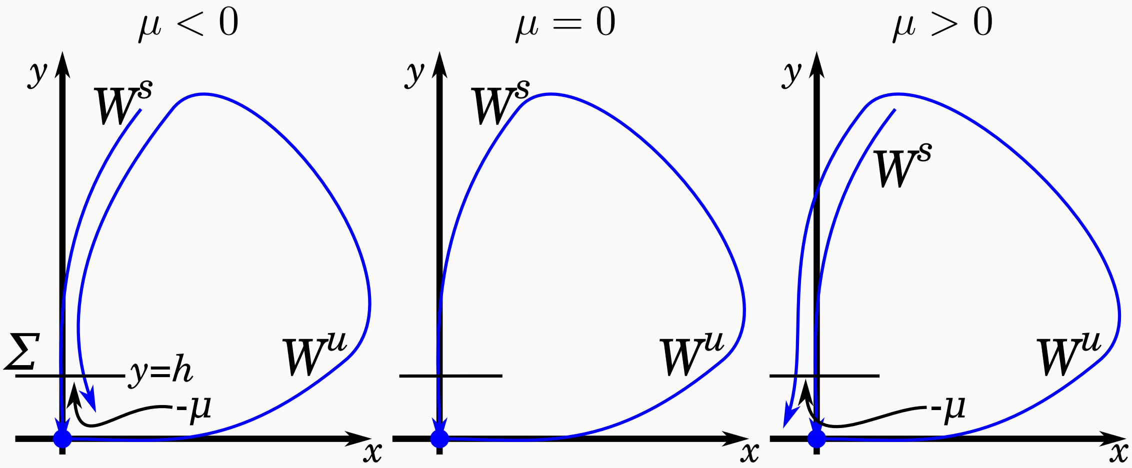
Global bifurcations VII



With $\mu < 0$, W^u is tucked inside W^s; with $\mu = 0$, W^u and W^s coincide and there is a homoclinic orbit; with $\mu > 0$, W^u comes outside W^s.

There is necessarily an equilibrium point inside the region enclosed by the homoclinic orbit.

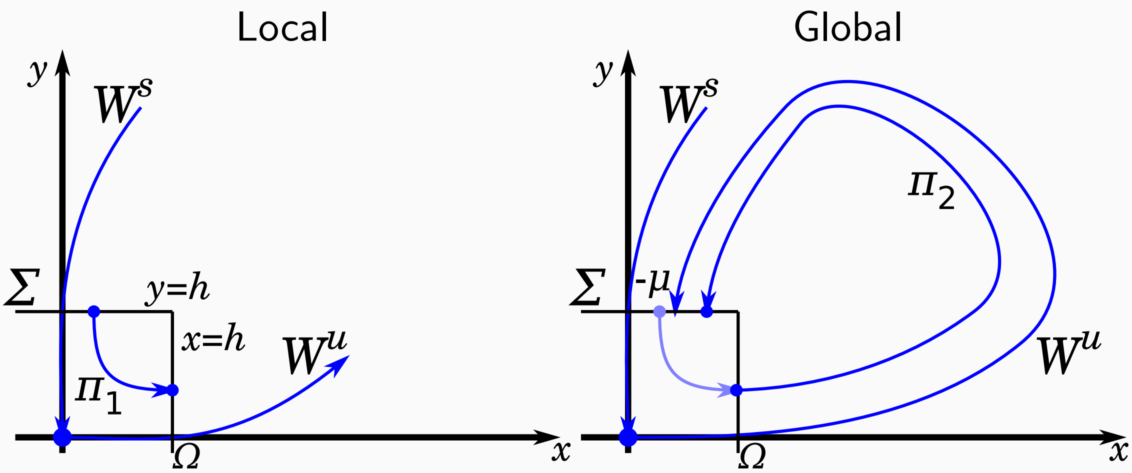
Global bifurcations VIII



We choose a codimension-one Poincaré section $\Sigma : y = h$ $(h \ll 1)$. Σ is transversal to W^s close to the equilibrium point.

We choose our parameter μ such that $-\mu$ measures the x coordinate of the point where W^u first intersects Σ as it approaches the equilibrium point. $-\mu$ is called the split function, and $\mu = 0$ is the codimension-one bifurcation condition.

Global bifurcations IX



We divide the phase space into two regions:

- Close to x^*, let $\pi_1 : \Sigma \to \Omega$ (Ω is the section $x = h$).
- Close to W^u, let $\pi_2 : \Omega \to \Sigma$

We will compose these local and global maps to get the Poincaré map $P = \pi_2 \circ \pi_1 : \Sigma \to \Sigma$.

Global bifurcations X

Recall the Hartman–Grobman theorem: the nonlinear phase portrait near a hyperbolic equilibrium point is topologically equivalent to the phase portrait of the linearised system. In fact, there is a (continuously differentiable) change of coordinates from the original (x, y) to a new system (we will use the same symbols (x, y)) such that near x^*, we have

$$\dot{x} = \lambda_+ x$$
$$\dot{y} = \lambda_- y$$

where $\lambda_+ > 0 > \lambda_-$ are the eigenvalues of the equilibrium point. Trajectories start on Σ at a point (x_0, h) at time $t = 0$, and they evolve according to:

$$x(t) = x_0 \exp(\lambda_+ t)$$
$$y(t) = h \exp(\lambda_- t).$$

Global bifurcations XI

Trajectories reach Ω at a time $t = T$, when $x(T) = h$, so $x_0 \exp(\lambda_+ T) = h$. Solving this yields

$$T = \frac{1}{\lambda_+} \log\left(\frac{h}{x_0}\right).$$

At this time, the y coordinate is $y_1 = y(T)$:

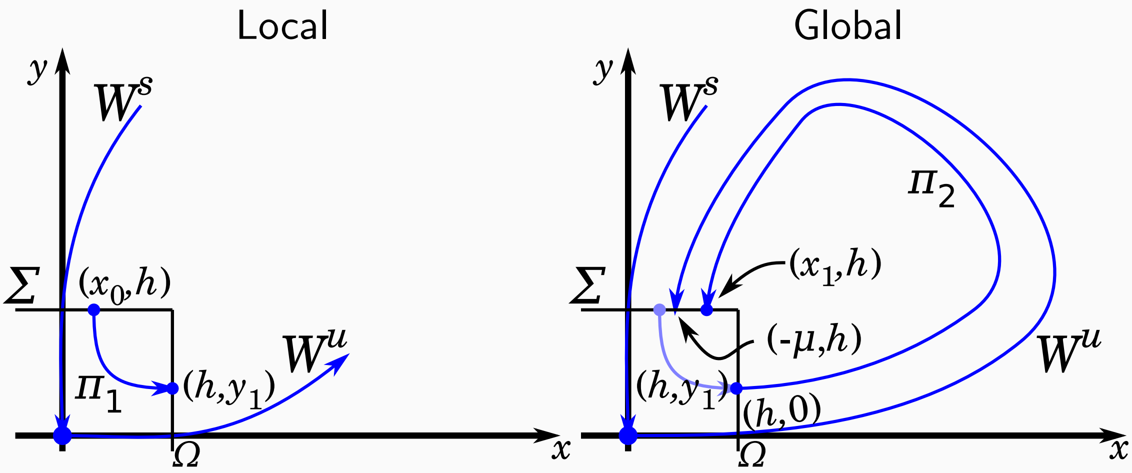
$$y_1 = h \exp(\lambda_- T) = h \left(\frac{h}{x_0}\right)^{\frac{\lambda_-}{\lambda_+}} = h \left(\frac{x_0}{h}\right)^{\delta},$$

where $\delta = -\lambda_-/\lambda_+$ is called the saddle index. Thus we have the map $\pi_1 : \Sigma \to \Omega$:

$$\pi_1 : (x_0, h) \to \left(h, y_1 = h \left(\frac{x_0}{h}\right)^{\delta}\right).$$

Note: the linear flow leads to a nonlinear map.

Global bifurcations XII



The global map π_2, close to W^u, takes us from Ω, starting at (h, y_1), to (x_1, h) on Σ.

The unstable manifold W^u leaves the neighbourhood of x^* when it crosses Ω at $(h, 0)$; it returns to Σ at $(-\mu, h)$, by the definition of the split function, which we take as our parameter.

Global bifurcations XIII

Trajectories depend smoothly on initial conditions (provided that W^u does not visit any other equilibria), and so x_1 is a smooth function of y_1, with $x_1 = -\mu$ when $y_1 = 0$.

Since we are close to W^u, we can Taylor expand this smooth function, keeping the first two terms:

$$\pi_2 : (h, y_1) \to \left(x_1 = -\mu + Ay_1 + \mathcal{O}(y_1^2), h\right),$$

where A is a constant that depends on the global properties of the flow. In planar systems, $A > 0$ since trajectories cannot cross. Note: the nonlinear flow leads to a linear map.

We now compose these to get the Poincaré map $P : \Sigma \to \Sigma$:

$$P : (x_0, h) \to \left(x_1 = -\mu + Ax_0^\delta + \mathcal{O}(x_0^{2\delta}), h\right),$$

where we have absorbed powers of h into A, and we will drop the higher order terms.

Global bifurcations XIV

This map can be iterated, so that after n circuits,

$$x_{n+1} = P(x_n) = -\mu + Ax_n^\delta = P(P(x_{n-1})) = \dots$$

or $x_n = P^n(x_0)$.
How does this help us understand global bifurcations? As usual with Poincaré maps, fixed points of the map P with small values of $x = x_\Gamma$ correspond to periodic orbits Γ of the flow. Fixed point satisfy

$$x_\Gamma = P(x_\Gamma) = -\mu + Ax_\Gamma^\delta,$$

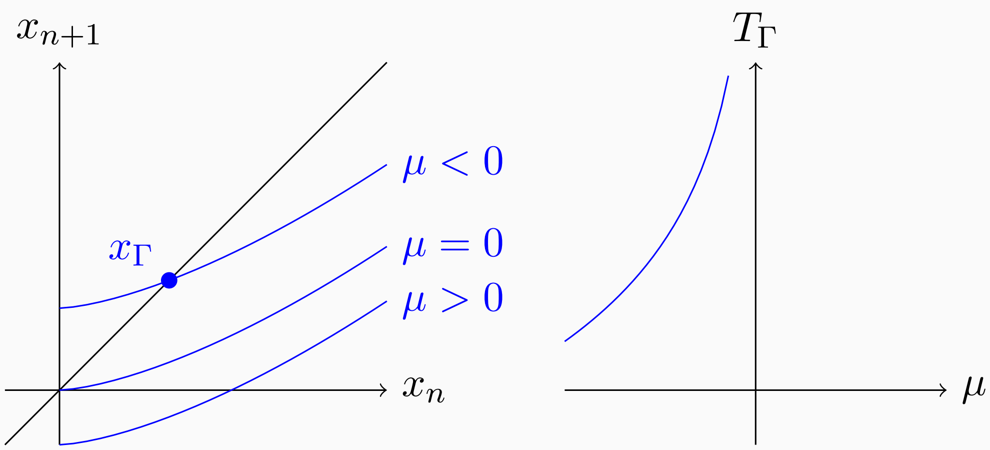
and the period of these orbits is close to

$$T_\Gamma = -\frac{1}{\lambda_+} \log x_\Gamma,$$

since the period is dominated by the long time spent in the neighbourhood of x^*.

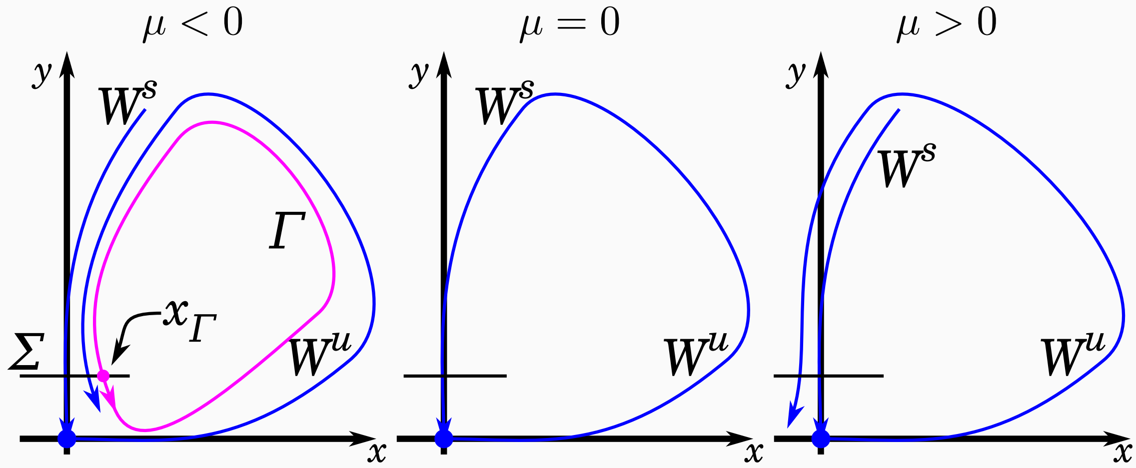
Global bifurcations XV

First, consider $\delta > 1$: $x_{n+1} = -\mu + Ax_n^\delta$ looks like a parabola.



For $\delta > 1$, the fixed point for small x is approximately $x_\Gamma = -\mu$. This corresponds to a stable periodic orbit since $|P'(x_\Gamma)| < 1$; this orbit exists for $\mu < 0$ and is destroyed in the homoclinic bifurcation at $\mu = 0$.

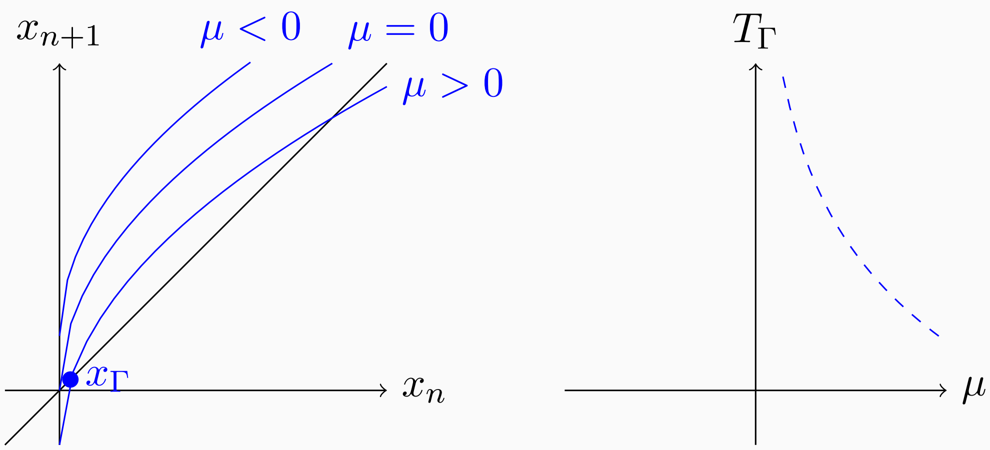
Global bifurcations XVI



The stable periodic orbit Γ exists for $\mu < 0$. For $\mu > 0$, all trajectories escape from the neighbourhood of x^*.

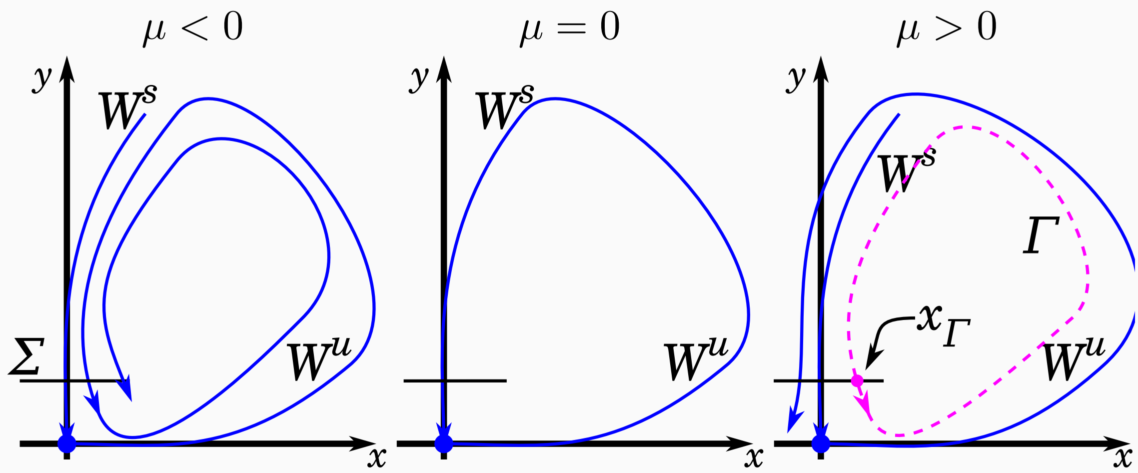
Global bifurcations XVII

Next, consider $\delta < 1$: $x_{n+1} = -\mu + Ax_n^\delta$ looks like a square root.



For $\delta < 1$, the fixed point for small x is approximately $x_\Gamma = (\mu/A)^{1/\delta}$. This corresponds to an unstable periodic orbit since $P'(x_\Gamma) > 1$; this orbit exists for $\mu > 0$ and is destroyed in the homoclinic bifurcation at $\mu = 0$.

Global bifurcations XVIII



The unstable periodic orbit Γ exists for $\mu > 0$. For all μ, almost all trajectories escape from the neighbourhood of x^*.

Note that the stability of the periodic orbit created in the global bifurcation, and the parameter range in which it exists depends on δ, the ratio of eigenvalues at the equilibrium point.

Global bifurcations XIX

Similar constructions work for $n = 3$ and higher, and they result in $n - 1$ dimensional Poincaré maps. In some cases, the dimension of the map can be reduced down to 1 or 2, and there is the possibility of finding chaotic dynamics. Two well known examples are the Lorenz map:

$$x_{n+1} = \mathsf{sgn}(x_n)\left(-\mu + A|x_n|^\delta\right)$$

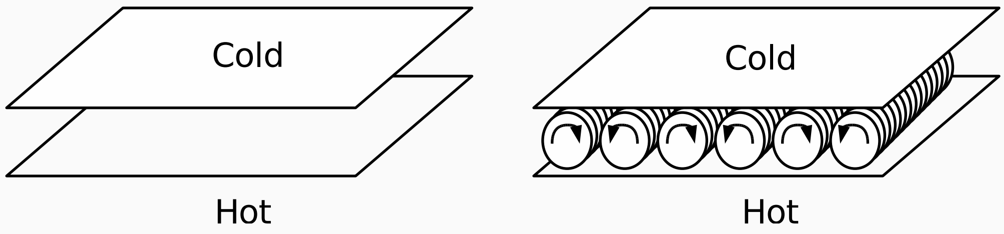
and the Shil'nikov map:

$$x_{n+1} = -\mu + Ax_n^\delta \cos(\gamma \log(x_n) + \Phi),$$

both of which can have infinitely many periodic points for $\delta < 1$.

PDEs, patterns, and the role of symmetry I

Convection:



- the layer of fluid is heated from below
- if the temperature difference is great enough, the fluid starts moving
- the warmer fluid expands and rises
- the colder fluid contracts and falls

PDEs, patterns, and the role of symmetry II

Irregular hexagonal patterns in heated cooking oil with turmeric:

- Put 1 mm cooking oil into a flat-bottomed pan
- Mix some turmeric into the oil – enough to finely coat the bottom of the pan
- Put the pan on a flat even heat source and heat very gently for a few seconds
- **Do not let the oil get very hot**

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Copyrighted figure omitted

Nick Safford (2004)

PDEs, patterns, and the role of symmetry III

Common features of many of these experiments:

- The system is isotropic in the two horizontal directions
- There is a trivial solution that becomes unstable once the forcing exceeds a critical value
- The pattern sets in with a preferred wavelength
- Initially the pattern grows without oscillations (though in other settings, waves are possible)
- All modes with wavelengths close to the preferred wavelength grow, regard of their orientation
- Nonlinear interactions determine which pattern is stable
- The symmetries of the patterns are very clear

Interesting patterns are formed close to onset, amplitudes are small, so analysis should be possible

PDEs, patterns, and the role of symmetry IV

The Swift–Hohenberg (1977) equation was originally written down to model the effects of thermal fluctuations close to the onset of convection. It illustrates many of the important issues in stability theory, and it is of interest as a generic pattern-forming PDE:

$$\frac{\partial U}{\partial t} = \mu U - (1 + \nabla^2)^2 U - U^3,$$

where $U(x, y, t)$ is a real scalar field representing the pattern, μ is a parameter that plays the role of a driving force, $(1 + \nabla^2)^2 U$ means $U + 2\nabla^2 U + \nabla^2\nabla^2 U$ ($\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$), and $(x, y) \in \mathbb{R}^2$.

Other choices of nonlinearity are possible.

As a model of convection, U represents the mid-layer temperature or vertical velocity, but the Swift–Hohenberg equation cannot be derived from the equations for convection in a systematic way.

PDEs, patterns, and the role of symmetry V

The Swift–Hohenberg equation has a trivial solution: $U = 0$. If U is close to zero, we can linearise:

$$\frac{\partial U}{\partial t} = \mu U - (1 + \nabla^2)^2 U = \mathcal{L}U.$$

This constant-coefficient linear PDE can be solved in terms of exponentials: write

$$U(x, y, t) = \exp(\sigma t) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}),$$

where σ is the growth rate, and $\exp(i\boldsymbol{k} \cdot \boldsymbol{x})$ is called a mode with wavevector $\boldsymbol{k} = (k_x, k_y)$. Putting this anzatz into the linearised PDE results in

$$\sigma e^{\sigma t + i\boldsymbol{k}\cdot\boldsymbol{x}} = \left(\mu - (1 - k^2)^2\right) e^{\sigma t + i\boldsymbol{k}\cdot\boldsymbol{x}} \quad \text{or} \quad \sigma = \mu - (1 - k^2)^2,$$

where $k = |\boldsymbol{k}|$ is the wavenumber of the mode.

PDEs, patterns, and the role of symmetry VI

The full linear solution is then the superposition of all modes:

$$U(x, y, t) = \iint_{-\infty}^{+\infty} \tilde{U}_0(k_x, k_y)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, e^{(\mu-(1-k^2)^2)t}\, dk_x dk_y,$$

where $\tilde{U}_0(k_x, k_y)$ is the Fourier transform of the initial condition $U(x, y, 0)$. The relation

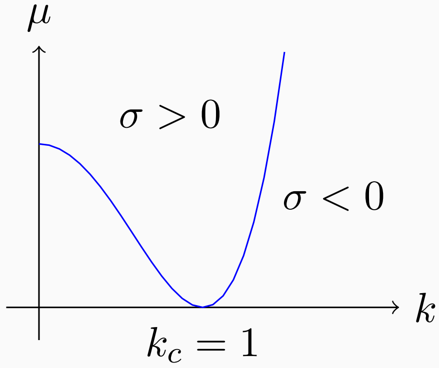
$$\sigma = \mu - (1 - k^2)^2$$

is called the dispersion relation: it relates the time-dependence (growth, decay or oscillation) of a mode to its wavenumber k.

In this example, there are no oscillations (σ is real), so σ is the growth rate of small disturbances with wavenumber k – and since σ depends on k, some of the components of the initial condition could grow while others decay.

PDEs, patterns, and the role of symmetry VII

We plot the curve $Re(\sigma) = 0$ in the (k, μ) plane, using the dispersion relation to relate k and μ: $\mu = (1 - k^2)^2$.

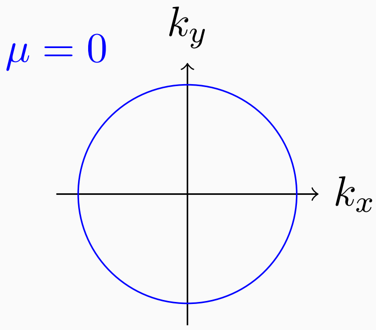


This is called the neutral stability curve: above the curve, modes are unstable, below the curve, modes are stable. The minimum of this curve is at $\mu = 0$ and wavenumber $k = k_c = 1$.

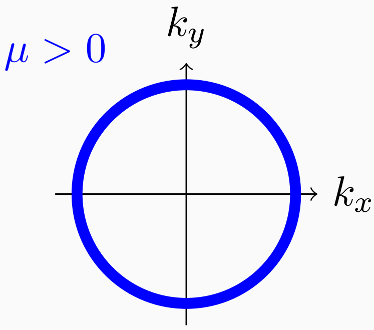
There is a pattern-forming instability as the parameter μ is increased through zero.

PDEs, patterns, and the role of symmetry VIII

In linear theory, all unstable modes grow exponentially without bound. Eventually, the amplitude will become large enough that the assumption of small amplitude no longer holds.



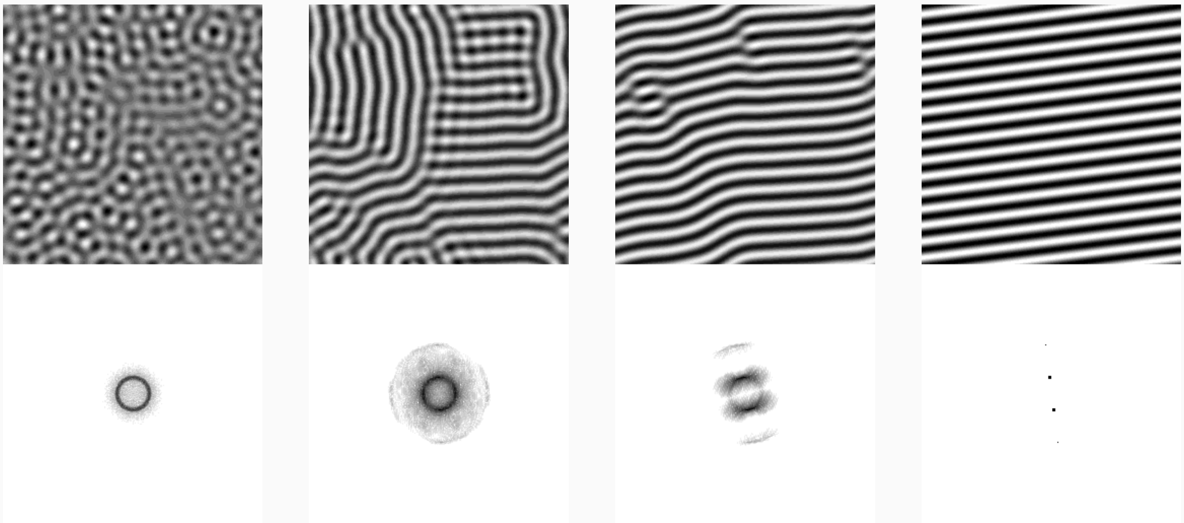
All modes on the critical circle $k = 1$ are marginally stable.



All modes in the annulus are unstable.

In order to predict the eventual amplitude or orientation of the modes, we turn to a nonlinear theory.

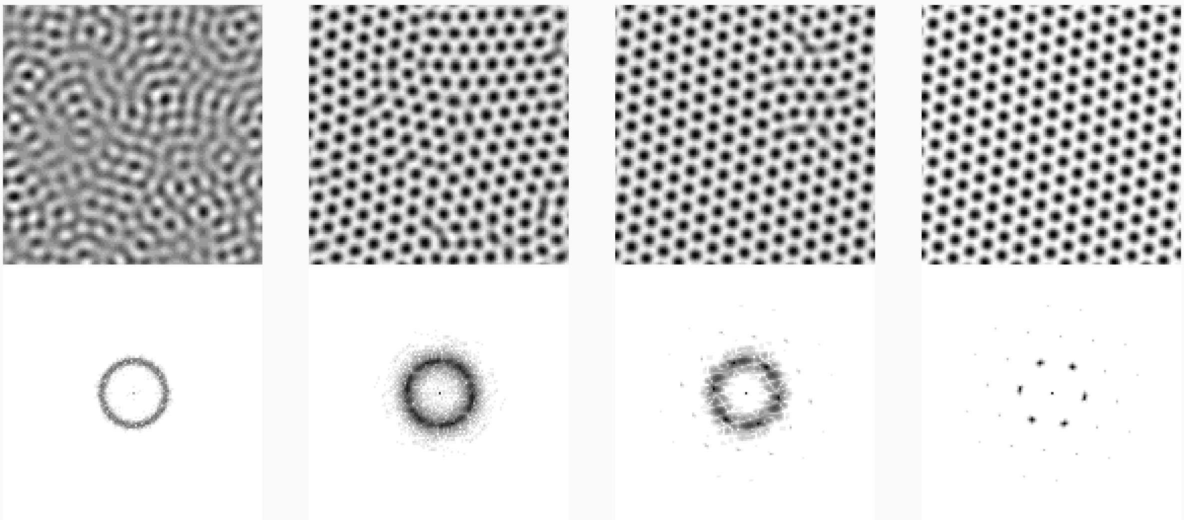
PDEs, patterns, and the role of symmetry IX



$$U_t = \mu U - (1 + \nabla^2)^2 U - U^3 \qquad \mu = 0.1$$

Domain: 16×16 with periodic boundary conditions. In the nonlinear phase, patches of stripes coarsen until only a few defects are left, the defects glide slowly, and annihilate in pairs, until a uniform pattern of stripes remains.

PDEs, patterns, and the role of symmetry X



$$U_t = \mu U - (1 + \nabla^2)^2 U + 0.5\, U^2 - U^3 \qquad \mu = 0.1$$

With a quadratic nonlinearity, hexagons are preferred.
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PDEs, patterns, and the role of symmetry XI

Let us consider the simplest possible case: the one-dimensional Swift–Hohenberg equation on an interval of length 2π, with periodic boundary conditions. We will look at the slow evolution of small amplitude solutions close to onset.

In one dimension, the PDE is:

$$\frac{\partial U}{\partial t} = \mu U - (1 + \partial_{xx})^2 U - U^3,$$

with boundary conditions:

$$U(x + 2\pi, t) = U(x, t).$$

We choose the length of the domain $L = 2\pi$ since $k_c = 1$.

PDEs, patterns, and the role of symmetry XII

We introduce three related small parameters:

- $U = \epsilon U_1 + \epsilon^2 U_2 + \epsilon^3 U_3 + \ldots$ — small amplitude plus nonlinear corrections
- $\frac{\partial}{\partial t} \rightarrow \epsilon^m \frac{\partial}{\partial t}$ — slow evolution
- $\mu = \mu_0 + \epsilon^n \mu_1$ — close to onset ($\mu_0 = 0$)

The constants $m > 0$ and $n > 0$ (to be determined) relate the smallness of the three small parameters.

μ_0 is the value of μ at which there is a marginally unstable mode ($\mu_0 = 0$ in this case).

PDEs, patterns, and the role of symmetry XIII

Substitute these scalings into the PDE:

$$\begin{aligned}
\epsilon^{1+m}\frac{\partial U_1}{\partial t} + \epsilon^{2+m}\frac{\partial U_2}{\partial t} + \epsilon^{3+m}\frac{\partial U_3}{\partial t} + \cdots = \\
\epsilon\left(-(1+\partial_{xx})^2\right)U_1 + \epsilon^2\left(-(1+\partial_{xx})^2\right)U_2 + \\
\epsilon^3\left(-(1+\partial_{xx})^2\right)U_3 + \cdots + \\
\epsilon^{1+n}\mu_1 U_1 + \epsilon^{2+n}\mu_1 U_2 + \epsilon^{3+n}\mu_1 U_3 + \dots \\
- \epsilon^3 U_1^3 - 3\epsilon^4 U_1^2 U_2 + \dots
\end{aligned}$$

With $\epsilon \ll 1$, $m > 0$ and $n > 0$, the largest term is $\mathcal{O}(\epsilon)$:

$$\mathcal{O}(\epsilon): \qquad 0 = -(1+\partial_{xx})^2 U_1.$$

PDEs, patterns, and the role of symmetry XIV

We can solve this by Fourier transform: write

$$U_1 = z(t)e^{ikx} + \bar{z}(t)e^{-ikx}$$

so that U_1 is real. The wavenumber k is an integer because of our choice of periodic domain. Then

$$0 = -(1-k^2)^2\left(z(t)e^{ikx} + \bar{z}(t)e^{-ikx}\right),$$

which only has a nontrivial solution when $k = 1$.

We define the singular linear operator $\mathcal{L}$:

$$\mathcal{L}U = -\left(1+\partial_{xx}\right)^2 U, \quad \text{with} \quad \mathcal{L}\,e^{ikx} = -(1-k^2)^2\,e^{ikx}.$$

PDEs, patterns, and the role of symmetry XV

Let us now examine the next largest terms in the PDE:

$$\epsilon^{1+m}\frac{\partial U_1}{\partial t} + \cdots = \epsilon^2 \mathcal{L} U_2 + \epsilon^3 \mathcal{L} U_3 + \epsilon^{1+n}\mu_1 U_1 - \epsilon^3 U_1^3 + \ldots$$

Within this equation, we can look at the e^{ix} component:

$$\epsilon^{1+m}\frac{dz}{dt} + \cdots = 0 + 0 + \epsilon^{1+n}\mu_1 z - 3\epsilon^3 |z|^2 z + \ldots$$

The zeroes appear since $\mathcal{L}e^{ix} = 0$, so the $\mathcal{L}U_2$ and $\mathcal{L}U_3$ terms have no e^{ix} component. In addition, we have used

$$\left(z(t)e^{ix} + \bar{z}(t)e^{-ix}\right)^3 = z^3 e^{3ix} + 3|z|^2 z e^{ix} + \text{c.c.}$$

PDEs, patterns, and the role of symmetry XVI

The only choice of m and n that makes sense is $m = n = 2$ (so that we keep the time dependence, the parameter and the nonlinearity), and so we get

$$\frac{dz}{dt} = \mu_1 z - 3|z|^2 z$$

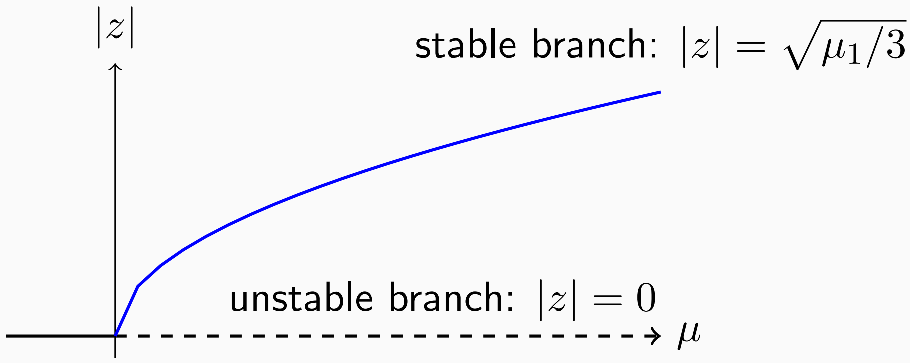
This is called the Landau equation and it is the normal form for a pitchfork bifurcation with rotation symmetry, or a pitchfork of revolution.

Other problems may require different scalings, but in almost all cases, we need time derivative, linear term and nonlinear term all of the same order of magnitude.

This procedure can be done more formally using the Fredholm alternative for the singular linear operator $\mathcal{L}$.

PDEs, patterns, and the role of symmetry XVII

The Landau equation has equilibrium solutions $z = 0$ and $z = \sqrt{\mu_1/3}\, e^{i\phi}$ for positive μ_1 and for arbitrary phase ϕ. The trivial solution is stable for $\mu < 0$ and unstable for $\mu > 0$; the nontrivial solution exists and is stable for $\mu > 0$ (exercise).



PDEs, patterns, and the role of symmetry XVIII

The equilibria of the Landau equation can be reconstituted to give approximate weakly nonlinear steady solutions to the PDE: with $z = \sqrt{\mu_1/3}\, e^{i\phi}$, we have

$$U_1 = \sqrt{\frac{\mu_1}{3}} \left(e^{i(x+\phi)} + e^{-i(x+\phi)} \right) = 2\sqrt{\frac{\mu_1}{3}} \cos(x+\phi).$$

But

$$U = \epsilon U_1 + \epsilon^3 U_3 + \dots \quad \text{and} \quad \mu = 0 + \epsilon^2 \mu_1$$

so $\epsilon\sqrt{\mu_1} = \sqrt{\mu}$ and

$$U(x) = \sqrt{\frac{4\mu}{3}} \cos(x+\phi) + \mathcal{O}(\mu^{3/2}).$$

The corrections can be calculated by solving the rest of the order ϵ^3 equation:

$$0 = \mathcal{L}U_3 + z^3 e^{3ix} + \bar{z}^3 e^{-3ix}.$$

PDEs, patterns, and the role of symmetry XIX

The role of symmetry: the original problem had reflection and translation symmetries: if $U(x)$ is a steady solution of the PDE, then $U(-x)$ and $U(x+\phi)$ are also steady solutions of the PDE (verify this by plugging these in to the PDE).

If we write $U(x) = ze^{ix} + \bar{z}e^{-ix}$, then

$$U(-x) = \bar{z}e^{ix} + ze^{-ix} \quad \text{and} \quad U(x+\phi) = (ze^{i\phi})e^{ix} + \overline{(ze^{i\phi})}e^{-ix},$$

so the action of the reflection is to take $z \to \bar{z}$ that of the translation is to take $z \to ze^{i\phi}$. The normal form is equivariant with respect to these actions: if $\dot{z} = f(z) = \mu_1 z - 3|z|^2 z$, then

$$f(\bar{z}) = \overline{f(z)} \quad \text{and} \quad f(ze^{i\phi}) = f(z)e^{i\phi}.$$

MAGIC046 covers equivariant bifurcation theory in detail.

PDEs, patterns, and the role of symmetry XX

The reduction from an infinite-dimensional PDE to a finite-dimensional ODE is justified by the Centre Manifold Theorem for PDEs (see for example, Carr 1981).

- For $\mu = \mu_0$, N eigenvalues have zero real part, and the span of the corresponding eigenvectors (modes) is called the linear centre manifold
- If there is a spectral gap then the solution of the PDE can be written as

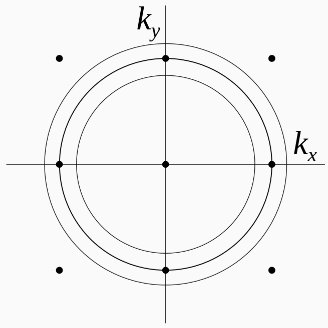
$$U(x,y,t) = \sum_{j=1}^{N} z_j(t) \exp(i\boldsymbol{k}_j \cdot \boldsymbol{x}) + \Phi(z_1, \ldots, z_N)$$

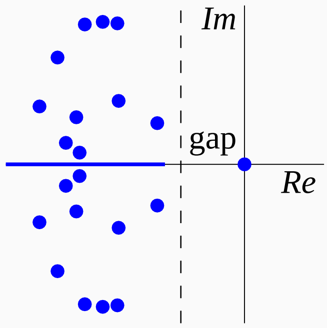
  $\boldsymbol{k}_j$ are wavevectors, $z_j(t)$ are the amplitudes of modes, and Φ includes the effects of all the damped modes

PDEs, patterns, and the role of symmetry XXI

Spectral gap.

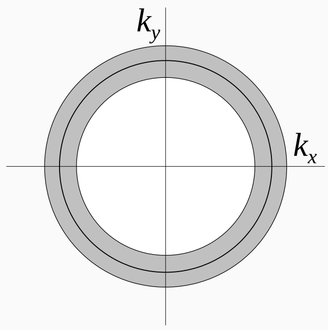
In a periodic domain, only a lattice of wavevectors is permitted

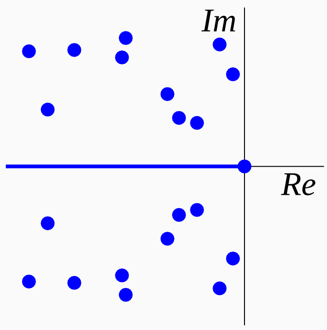




No spectral gap.

With $(x, y) \in \mathbb{R}^2$, all modes in the annulus are unstable





Numerical methods for dynamical systems I

Important topics:

- Symbolic algebra: equilibrium points, bifurcations. Packages include Maple, matlab (symbolic toolkit), Reduce.
- Integrating ODEs at single parameter values. Packages include matlab, IDL, python (SciPy).
- Continuation and the implicit function theorem.
- Continuation of local bifurcation points. Packages include matcont, AUTO, python (SciPy).
- Continuation of global bifurcation points. Packages include AUTO/Homcont. (Not enough time for this.)

Open source alternatives to matlab, IDL and Maple: octave (`http://www.octave.org`), gdl (`http://gnudatalanguage.sourceforge.net/`) and symbolic algebra package Reduce (`http://www.reduce-algebra.com`).

Numerical methods for dynamical systems II

General references and online sources:

- SIAM DSweb: Dynamical systems software `http://www.dynamicalsystems.org/sw/sw/`
- Kuznetsov *Elements of Applied Bifurcation Theory* Springer 2004 (see also `http://www.staff.science.uu.nl/~kouzn101/NBA/index.html`)
- Parker & Chua *Practical Numerical Algorithms for Chaotic Systems* Springer 1989
- SciPy (Scientific Python, `http://www.scipy.org`) has lots of built-in functions as well as a variety of useful packages, for example, PyDSTool (`http://pydstool.sourceforge.net/`).

Symbolic algebra I

Example system: the Lorenz equations:

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = rx - y - xz, \qquad \dot{z} = -bz + xy,$$

with $r > 0$, $\sigma > 0$ and $4 > b > 0$. We will use Reduce to find the local bifurcations (input in red, output in blue).

Set up the ODEs and solve for the equilibria:

```
xdot:=sig*(y-x);
ydot:=r*x-y-x*z;
zdot:=-b*z+x*y;
eqms:=solve({xdot,ydot,zdot},{x,y,z});
```

```
eqms:=
{{x=sqrt(b)*sqrt(r - 1), y=sqrt(b)*sqrt(r - 1), z=r-1},
 {x=-sqrt(b)*sqrt(r - 1), y=-sqrt(b)*sqrt(r - 1), z=r-1},
 {x=0,y=0,z=0}}
```

Symbolic algebra II

Identify the trivial and nontrivial equilibria:

```
eqm_triv:=third(eqms);
eqm_nontriv:=first(eqms);
```

```
eqm_triv := {x=0,y=0,z=0}
eqm_nontriv := {x=sqrt(b)*sqrt(r - 1),
                y=sqrt(b)*sqrt(r - 1),
                z=r - 1}
```

Work out the Jacobian matrix:

```
jac:=mat((df(xdot,x), df(xdot,y), df(xdot,z)),
         (df(ydot,x), df(ydot,y), df(ydot,z)),
         (df(zdot,x), df(zdot,y), df(zdot,z)));
```

Symbolic algebra III

```
       [ - sig  sig   0  ]
       [                 ]
jac := [r - z   -1    - x]
       [                 ]
       [  y      x    - b]
```

Jacobian for the trivial solution:

```
jac_triv:=sub(eqm_triv,jac);
```

```
            [ - sig  sig   0  ]
            [                 ]
jac_triv := [  r     -1    0  ]
            [                 ]
            [  0      0    - b]
```

Symbolic algebra IV

The trivial solution has one eigenvalue equal to $-b$, and the 2×2 part has trace and determinant:

```
det(jac_triv)/(-b);
trace(jac_triv) + b;

sig*( - r + 1)
 - (sig + 1)
```

so the trace is negative (no Hopf bifurcations possible) and the determinant is zero only when $r = 1$ (since $\sigma \neq 0$). The zero eigenvalue at $r = 1$ corresponds to the pitchfork bifurcation that creates the two nontrivial equilibria.

Symbolic algebra V

Jacobian for the nontrivial solution:

```
jac_nontriv:=sub(eqm_nontriv,jac);

jac_nontriv:=
[       - sig                sig                    0           ]
[                                                               ]
[         1                  -1            - sqrt(b)*sqrt(r-1)]
[                                                               ]
[sqrt(b)*sqrt(r-1)  sqrt(b)*sqrt(r-1)                - b        ]
```

Symbolic algebra VI

Characteristic equation for the nontrivial solution:

```
id3:=mat((1,0,0),(0,1,0),(0,0,1));
factor lamda;
char_eqn_nontriv:=det(lamda*id3 - jac_nontriv);
```

```
     3         2
lamda  + lamda *(b+sig+1) + lamda*b*(r+sig) + 2*b*sig*(r-1)
```

We get a zero eigenvalue when $\lambda = 0$ is a root of the characteristic equation, which implies $2b\sigma(r-1) = 0$, or $r = 1$ (since $\sigma \neq 0$ and $b \neq 0$).

Symbolic algebra VII

We look for a Hopf bifurcation with $\lambda = \pm i\omega$ as a pair of roots (with a real third root R), which implies that the characteristic equation factorises:

$$(\lambda^2 + \omega^2)(\lambda - R) = 0, \quad \text{or} \quad \lambda^3 - R\lambda^2 + \omega^2\lambda - R\omega^2 = 0.$$

Matching coefficients, we get:

```
rr:=-coeffn(char_eqn_nontriv,lamda,2);
om2:=coeffn(char_eqn_nontriv,lamda,1);
rrom2:=-coeffn(char_eqn_nontriv,lamda,0);
```

```
rr :=  - (b + sig + 1)
om2 := b*(r + sig)
rrom2 := 2*b*sig*( - r + 1)
```

Symbolic algebra VIII

Equating $R\omega^2 = R \times \omega^2$ and solving for r yields:

```
rhopf:=solve(rrom2-rr*om2,r);
```

```
               sig*( - b - sig - 3)
rhopf:= {r=----------------------}
                   b - sig + 1
```

or

$$r_{\mathsf{Hopf}} = \frac{\sigma(b+\sigma+3)}{\sigma-b-1}$$

provided $\omega^2 = b(r+\sigma) > 0$, or:

```
sub(rhopf,om2);
```

```
  - 2*b*sig*(sig + 1)
----------------------
     b - sig + 1
```

so we need $\sigma > 1+b$ (or $r > 0$) for the Hopf bifurcation to exist.

Integrating ODEs I

A variety of ODE integrators can be used, for example, python (SciPy). This file is in `lorenz.py` and can be run as
`python lorenz.py`
(There is an Octave version on
`http://en.wikipedia.org/wiki/Lorenz_attractor`)

```
from scipy import integrate
from pylab import *

# Definition of ODEs
def Lorenz(xyz, t, s, r, b):
    x, y, z = xyz
    return array([ s*(y - x), r*x - y - x*z, -b*z + x*y ])

s = 10.0                          # Parameters
r = 28.0
```

Integrating ODEs II

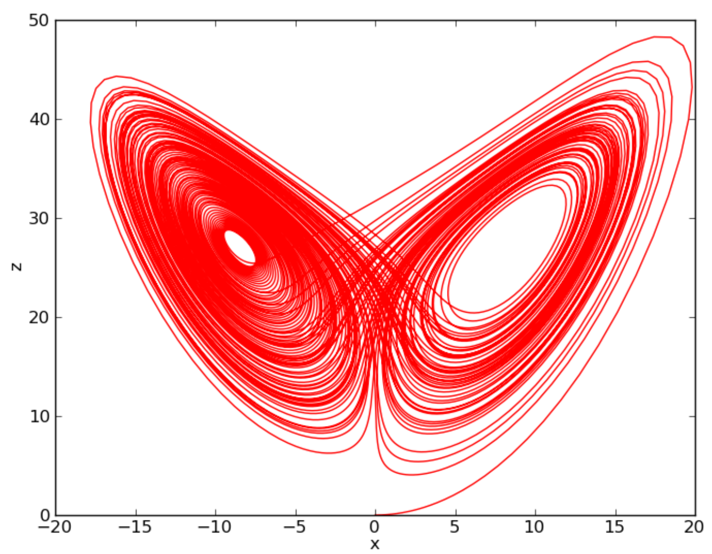
```
b = 8.0/3.0

time = linspace(0, 100,  10000)    # time
xyz0 = array([0.01, 0.01, 0.00])   # initials conditions

trajectory = integrate.odeint(Lorenz, xyz0, time,
             args=(s, r, b))

# We can now use Matplotlib to plot the evolution
x, y, z = trajectory.T
f1 = figure()
plot(x, z, 'r-')
xlabel('x')
ylabel('z')
f1.savefig('lorenz_xz.png')
show()
```

Integrating ODEs III



$r = 28$, $\sigma = 10$, $b = 8/3$, $r_{\text{Hopf}} = 24.737$

See also Michael Cross (Caltech): applet for the Lorenz equations

`http://crossgroup.caltech.edu/chaos_new/Lorenz.html`

Continuation: the implicit function theorem I

So far we have looked at computing equilibria "by hand" (disadvantage: can only be used for relatively simple systems) and solving the equations numerically (disadvantage: can only find stable solutions, and offers no explanation of what is going on). Suppose we are solving

$$\dot{x} = f(x;\mu) \quad \text{with} \quad f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,$$

and we know that at $\mu = \mu^*$, there is an equilibrium point $x = x^*$. We want to calculate the surface of equilibria $x_{eq}(\mu)$ that solves

$$f(x_{eq}(\mu);\mu) = 0,$$

with $x_{eq}(\mu^*) = x^*$. We also want to locate the bifurcations from this equilibrium point, and track how these bifucations points depend on the parameters.

Continuation: the implicit function theorem II

These can both be done using the Implicit Function Theorem and the idea of continuation, where we start at a known equilibrium point and calculate new equilibria for nearby parameter values.

Implicit Function Theorem: Suppose $f(x^*;\mu^*) = 0$ for some (x^*, μ^*), with f a smooth function from $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and suppose

$$J(x^*;\mu^*) = \left.\frac{\partial f}{\partial x}\right|_{x=x^*}$$

is nonsingular, then there is locally a unique smooth function $x_{eq}(\mu)$ such that $f(x_{eq}(\mu);\mu) = 0$.

Note: the parameter values at which the Implicit Function Theorem fails to provide a unique surface of equilibria are precisely those at which the Jacobian matrix has a zero eigenvalue, where there are saddle-node, transcritical or pitchfork bifurcations.

Continuation: the implicit function theorem III

Numerically, we focus on $\mu \in \mathbb{R}$ ($m = 1$), start with an equilibrium $x_0(\mu_0)$, and aim to compute a new equilibrium point at $\mu_1 = \mu_0 + \delta\mu$. A simple predictor–corrector algorithm is:

- Predictor step: guess that the new equilibrium will be at $x_1^p = x_0$.
- Corrector step: solve $f(x; \mu_1) = 0$ using Newton's method, with x_1^p as our initial guess, obtaining x_1 as the new equilbrium point.

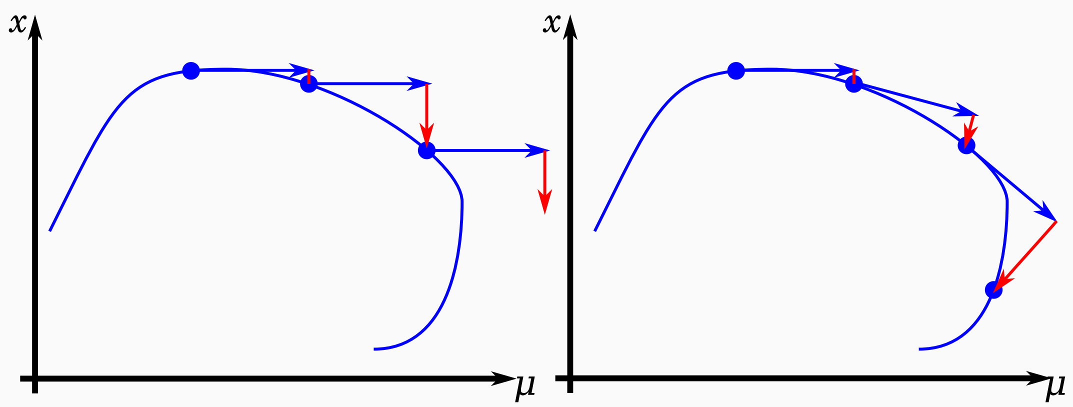
Recall: Newton's method involves looping:

$$x \to x - [J(x; \mu_1)]^{-1} f(x; \mu_1)$$

until convergence, and so may fail to converge to x_1 if either $J(x_1)$ is singular (same problem as the Implicit Function Theorem) or if x_1^p is not close enough to x_1 (this can be improved by using a smaller $\delta\mu$).

Continuation: the implicit function theorem IV

Pseudo-arclength continuation offers a way around this: let the curve of equilibria be parameterised by s, so we are seeking $(x(s), \mu(s))$ such that $(x(0), \mu(0)) = (x_0, \mu_0)$ and $f(x(s); \mu(s)) = 0$. We keep track of the tangent vector to the curve of equilibria, take the prediction step along the tangent vector at (x_0, μ_0), and take the Newton correction steps orthogonal to the tangent vector. With this method, curves of equilibria can be followed around saddle-node bifurcations.



Continuation: the implicit function theorem V

We can also continue loci of bifurcation points in a second parameter by augmenting our function f. For example, let

$$g(x; \mu_1, \mu_2) = (f(x; \mu_1, \mu_2), \det(J(x; \mu_1, \mu_2))),$$

with $g : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^{n+1}$. If we let $y = (x, \mu_1)$, then folowing curves of saddle-node bifurcations in $f(x; \mu_1, \mu_2)$ is equivalent to following curves of equilibria in $g(y; \mu_2)$, with μ_2 as the continuation parameter.

In fact there are better ways of doing this, and the idea can be extended to Hopf bifurcations.

Periodic orbits can be treated as boundary value problems; similar pseudo-arclength continuation methods are possible.

Continuation: local bifurcations I

AUTO (`http://cmvl.cs.concordia.ca/auto/`) is probably the most comprehensive continuation package, though others (e.g., matcont `http://sourceforge.net/projects/matcont/`) are probably easier to use.

An example: Circadian rhythm in Neurospora (Gonze *et al.*, 2000):

$$\begin{aligned}
\dot{M} &= v_s \frac{K_1^n}{K_1^n + F_N^n} - v_m \frac{M}{K_m + M}, \\
\dot{F}_C &= k_s M - v_d \frac{F_C}{K_d + F_C} - k_1 F_C + k_2 F_N, \\
\dot{F}_N &= k_1 F_C - k_2 F_N,
\end{aligned}$$

where M is the concentration of the FRQ mRNA, and F_C and F_N are concentrations of the cytosolic and nuclear forms of FRQ. The principal parameter (PAR(1)) is v_s, rate of FRQ transcription.

Continuation: local bifurcations II

For $v_s = 0$ there is a fixed point $M = F_C = F_N = 0$. We increase v_s and see what happens.

```
Starting neuro ...

  BR    PT  TY LAB    PAR(1)        L2-NORM         U(1)          U(2)
   1     1  EP   1   0.00000E+00   0.00000E+00   0.00000E+00   0.00000E+00
   1    87  HB   2   6.03700E-01   2.77826E+00   2.47085E+00   9.75865E-01
   1   100  EP   3   1.05874E+00   3.08246E+00   2.55063E+00   1.32968E+00

neuro ... done
```

AUTO detects a Hopf bifurcation (HB), so periodic solutions are created, which can also be continued:

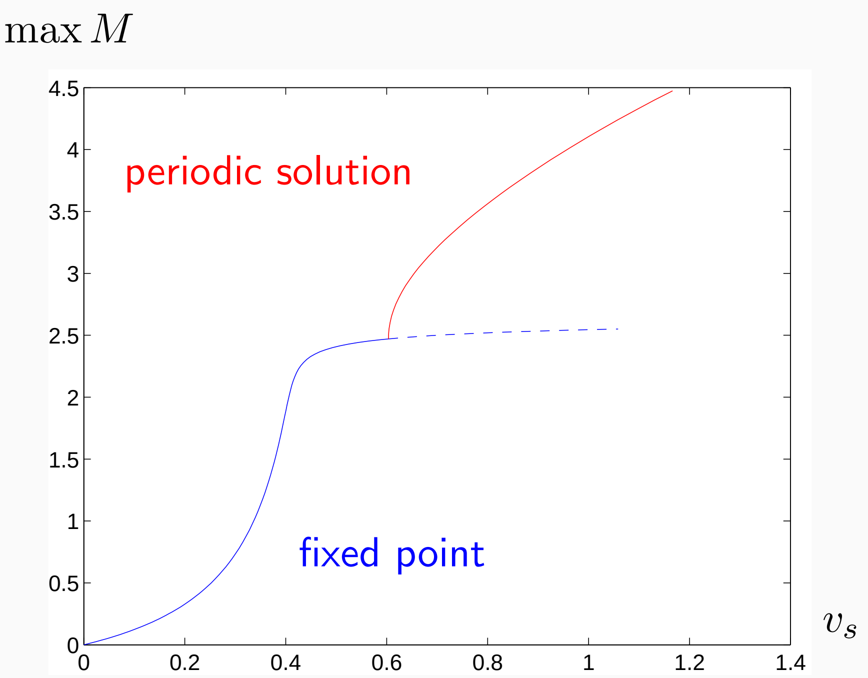
```
Starting neuro ...

  BR    PT  TY LAB    PAR(1)        L2-NORM        PERIOD
   2    99  UZ   4   2.18355E+00   6.36032E+00   2.40000E+01
   2   120  EP   5   2.66998E+00   7.27214E+00   2.57823E+01

neuro ... done
```

Continuation: local bifurcations III

Bifurcation diagram:



Continuation: local bifurcations IV

A particular solution, with period $T = 24$: