



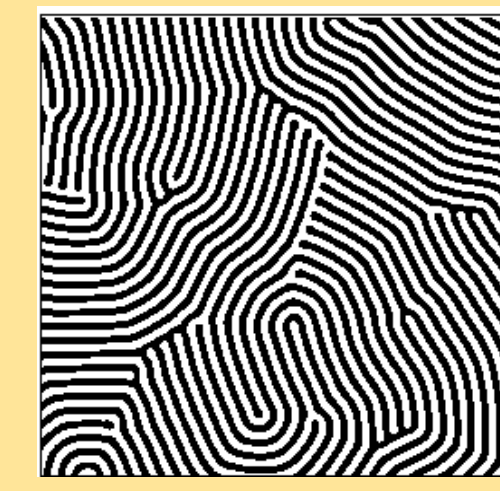
# FRONT-DYNAMICS AND PATTERN FORMATION IN THE WAKE OF TRIGGERED INSTABILITIES

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## How can external mechanisms be used to harness and control natural pattern forming processes?

### Pattern formation in nature

Creation of patterns from both localized perturbations and random fluctuations of an unstable state has been very well studied.



#### Downsides:

- Random initial data leads to formation of defects in patterns
- Uniform suppression of fluctuations needed to obtain defect-free pattern
- Patterns formed from local perturbations only depend on intrinsic system parameters

### Propagation into an unstable state

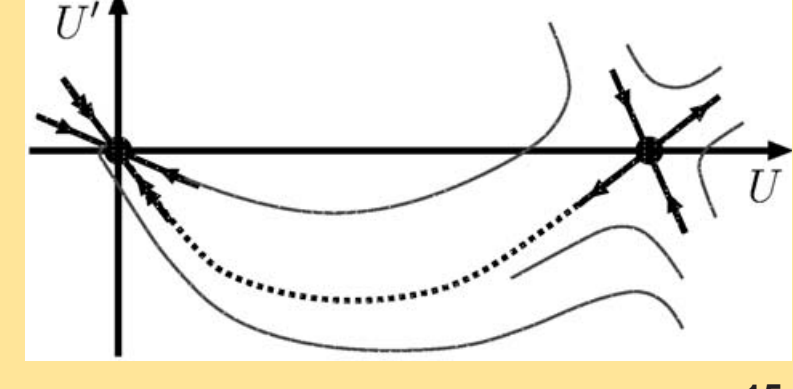
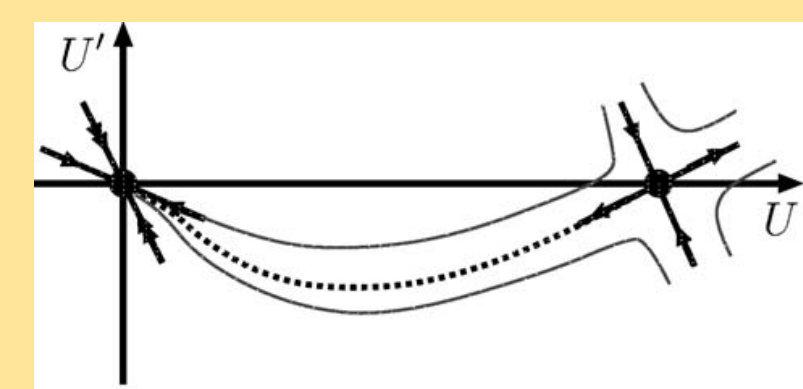
Pattern forming fronts come in two forms, corresponding to two types of free invasion fronts:

#### Pulled

- Local perturbation invades with speed  $c_{lin}$  predicted by linear dynamics
- Speed and wavenumber predicted by crossing of branch point of absolute spectrum

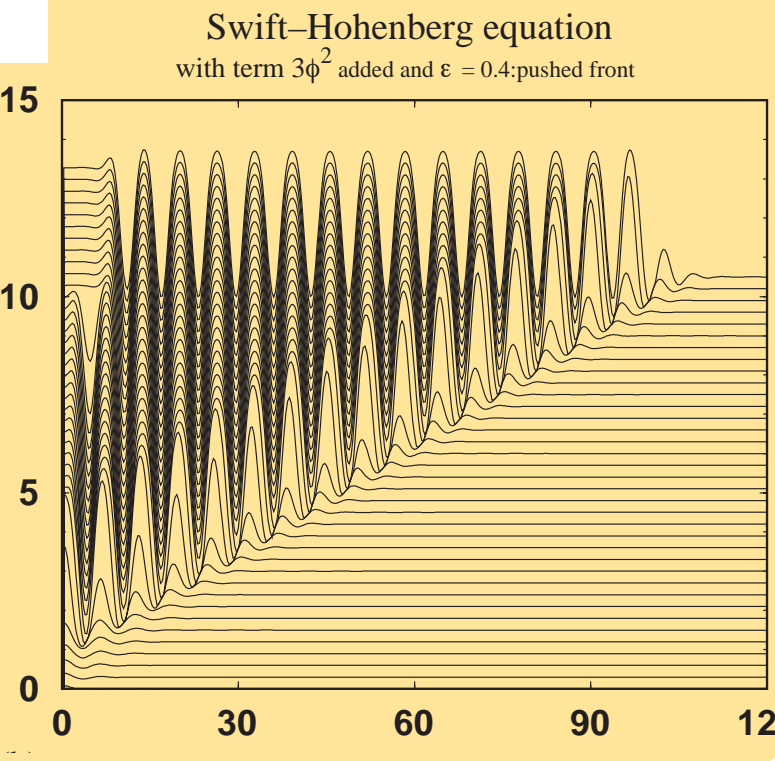
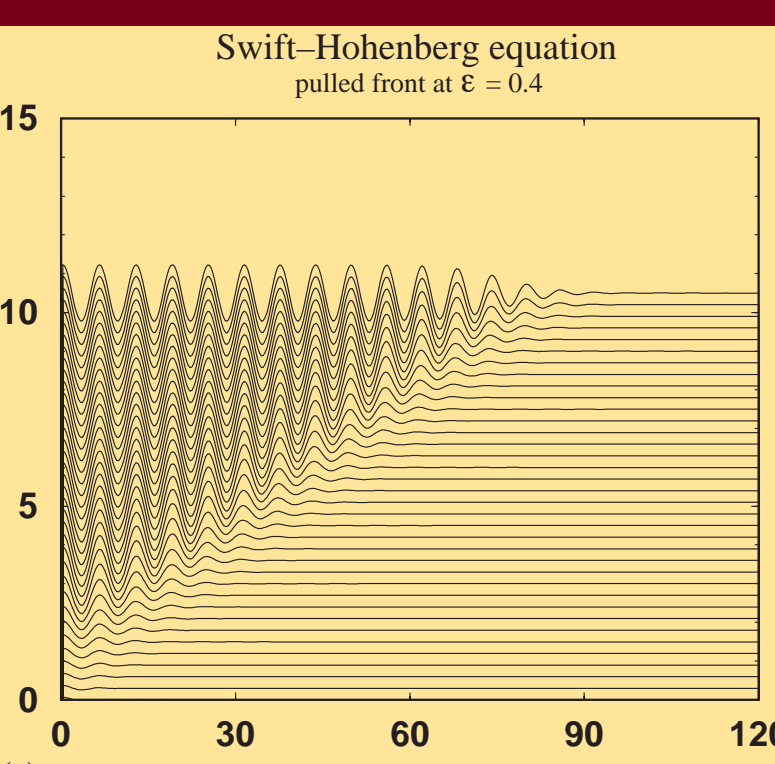
$$\Sigma_{abs} \subset \{\text{Re} \lambda \leq 0\}, \quad \Sigma_{abs} \cap i\mathbb{R} \neq \emptyset$$

- Fronts have leading order asymptotics  $(A_\infty e^{\nu_{ss}\xi} + B_\infty e^{\nu_{in}\xi})$  as  $\xi \rightarrow \infty$ ,  $\nu_{in}$  spatial eigenvalue corresponding to the branch point.



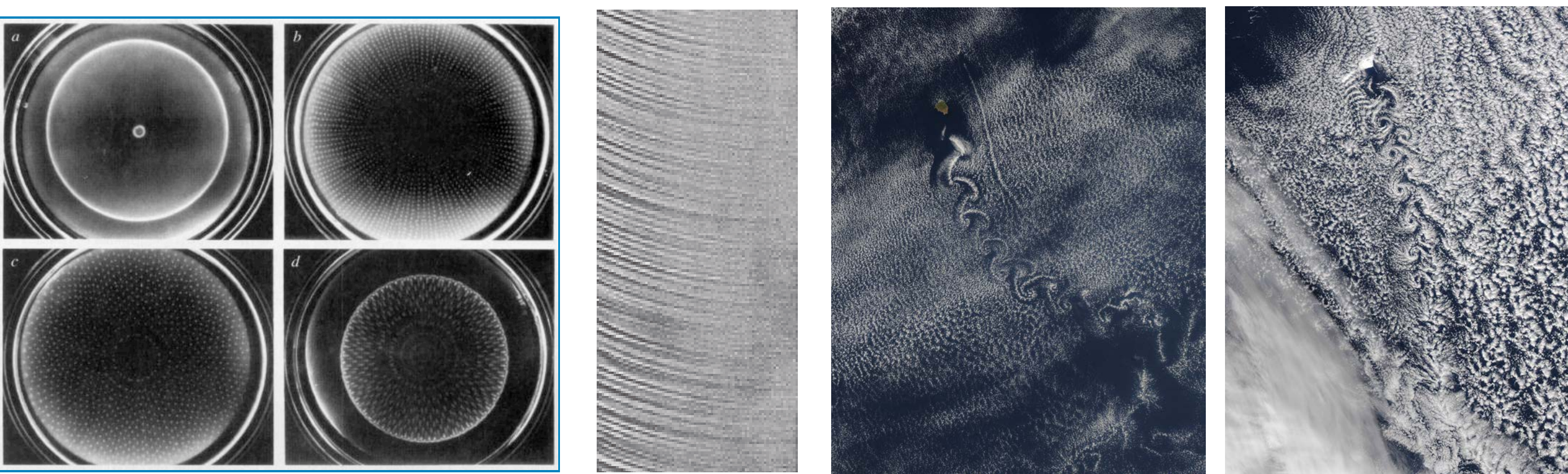
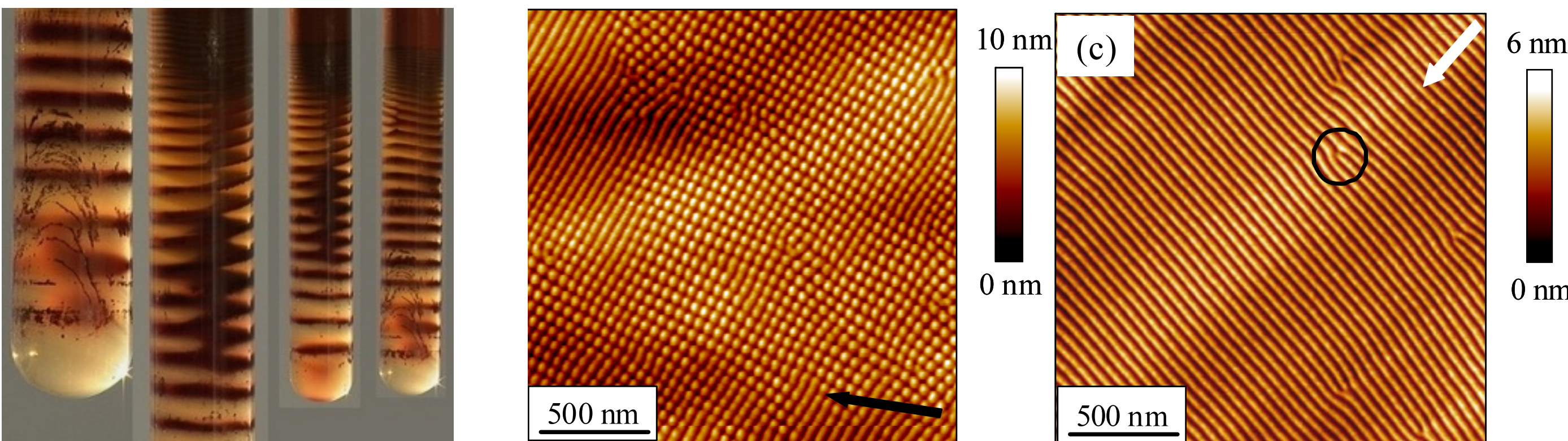
#### Pushed

- Invades with speed  $c_p > c_{lin}$ , arises from inherent nonlinearities of system
- Front profile has steep exponential tail, corresponding to approach along a strong-stable manifold
- Fronts generically possess oscillatory tails  $C_\infty e^{\nu_{ss}\xi} + c.c.$  as  $\xi \rightarrow +\infty$ , with  $\nu_{ss}$  a complex conjugate pair of strong-stable spatial eigenvalues.



### Triggered instabilities

- Begin with stable state  $\tilde{u}_*$ , use mechanism  $\chi$  to progressively excite, or *trigger*, system into an unstable state  $u_*$
- Patterns can then nucleate in the wake
- Select patterns by spatio-temporally controlling the excitation process



Figures from:  
• Gallo, Moro 2008  
• insilico.hu/iesegang  
• Ziberti 2009  
• Budrene, Berg 2005  
• Radon et. al 2004  
• Van Saarloos '04

• NASA AQUA MODIS 2012  
References:  
R. Goh, A. Scheel. Triggered Fronts in the Complex Ginzburg Landau Equation J. Nonlinear Science 24 (2014), 117-144.  
R. Goh, A. Scheel. Pattern formation in the wake of triggered pushed fronts, Manuscript.

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RG was supported by NSF grant GFRP-00006595

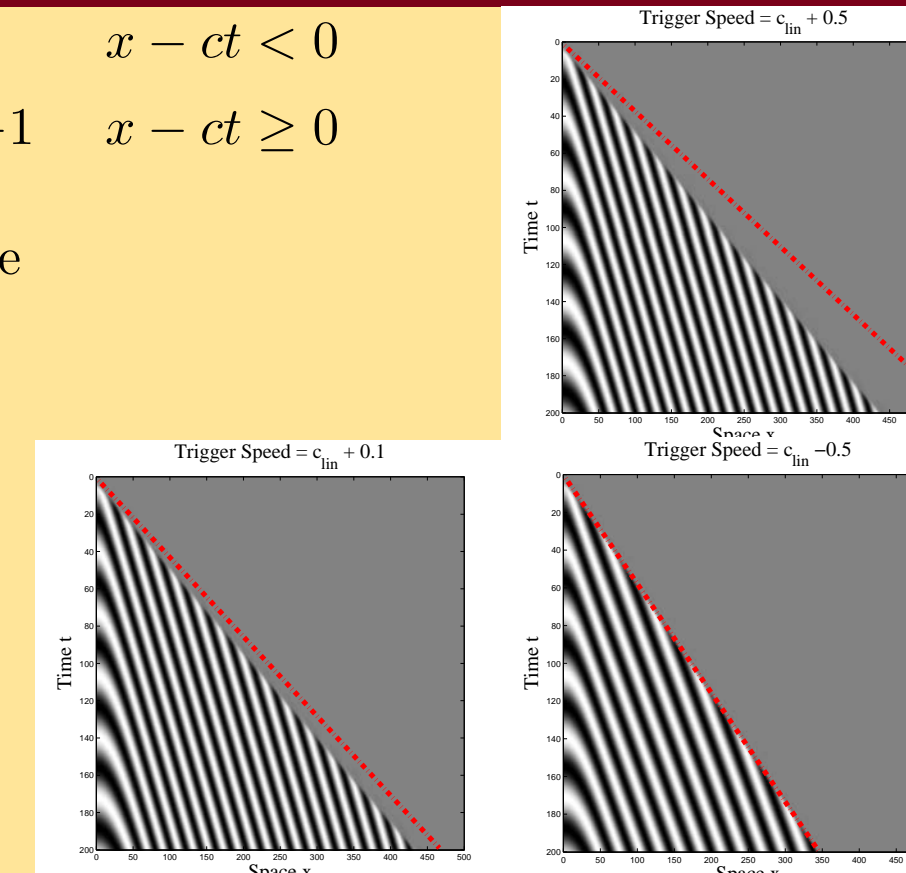
### Prototypical example: qCGL-equation

$$A_t = (1+i\alpha)A_{xx} + \chi A - (\mu+i\gamma)A|A|^2 - (\rho+i\beta)A|A|^4, \quad x, t \in \mathbb{R}, \quad \chi(x, t; c) = \begin{cases} 1 & x - ct < 0 \\ -1 & x - ct \geq 0 \end{cases}$$

- $\chi$  moves with speed  $c$ , changes stability of  $A_0 \equiv 0$  from stable to unstable
- For given speed  $c$  a *trigger front* is a solution  $A(x, t) = e^{i\omega t} A_{tf}(x - ct)$

$$A_{tf}(\xi) \rightarrow 0, \quad \xi \rightarrow \infty, \quad |A_{tf}(\xi) - re^{ik\xi}| \rightarrow \infty, \quad \xi \rightarrow -\infty.$$

- $k, \omega, c$  solve nonlinear dispersion relation  $1 = k^2 - \mu r^2 + r^2, \quad \omega + ck = \alpha k^2 - \gamma r^2 + \beta r^4.$

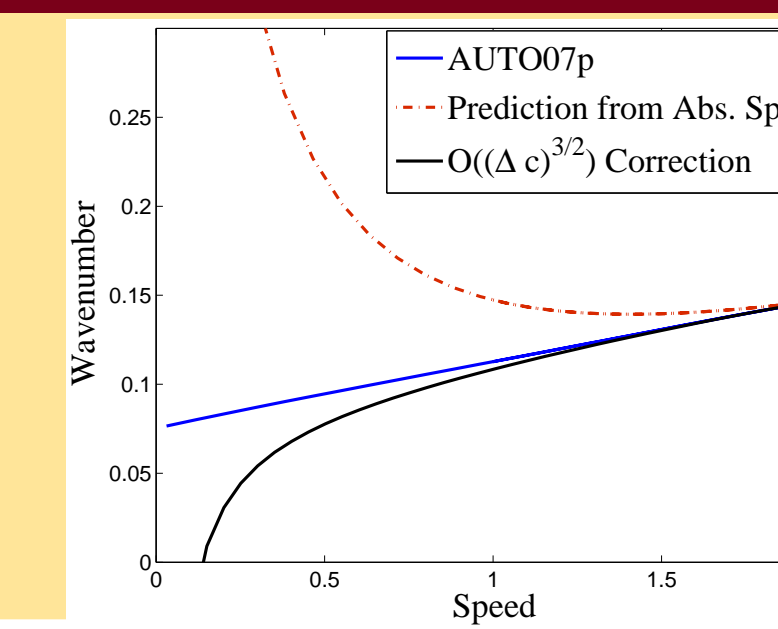


### Pulled case

**Theorem 1.** (RG, Scheel 14) Let  $\mu = 1, \rho = \beta = 0$ , and  $\alpha\gamma < 1$ . Under certain genericity conditions on free front, and  $0 < c_{lin} - c \ll 1$ , there exists a trigger front solution  $A_{tf}$  with angular frequency

$$\omega_{tf}(c) = \omega_{abs}(c) + \frac{2}{\pi}(1 + \alpha^2)^{\frac{3}{2}} |\Delta Z_i| (c_{lin} - c)^{\frac{3}{2}} + \mathcal{O}(c_{lin} - c)^2,$$

wavenumber  $k_{tf}(c)$  determined by dispersion relation,  $\Delta Z_i$  projective distance between  $T_0 W^{ss}(A_p)$  and  $T_0 W_+^s(0)$ ,  $\pm i\omega_{abs}(c) = \Sigma_{abs} \cap i\mathbb{R}$ .



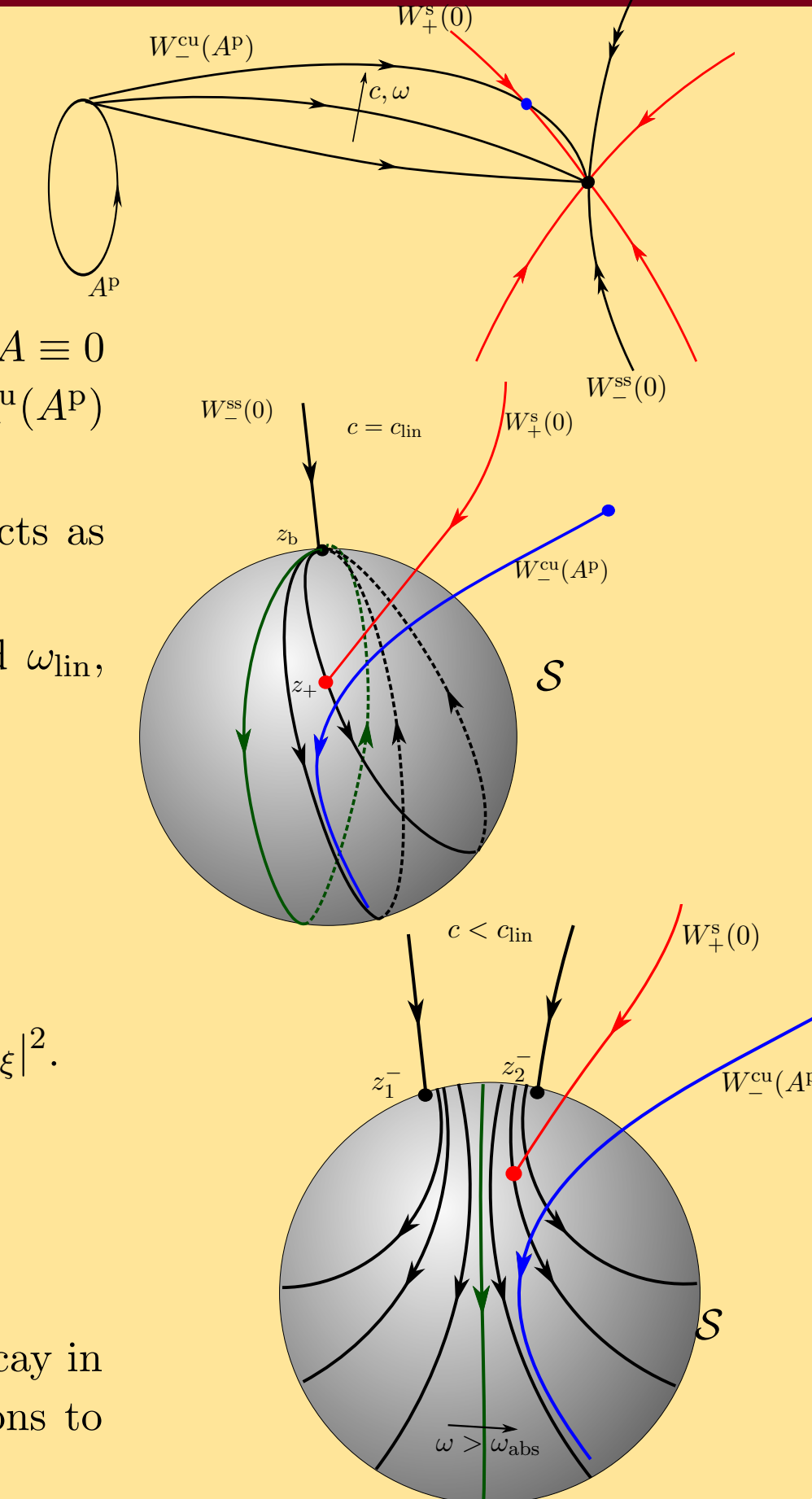
### Method of proof

#### Spatial Dynamics

- Detune  $A \rightarrow e^{i\omega t} A$ , co-moving frame  $\xi = x - ct$ :

$$A_\xi \xi = -\frac{1}{(1+i\alpha)} [(\chi(\xi) - i\omega)A + cA_\xi - (1 + \gamma i)A|A|^2].$$

- Perform a heteroclinic matching between the (relative) equilibria  $A^p$  and  $A \equiv 0$  to prove existence: i.e. find intersection between invariant manifolds  $W^{cu}(A^p)$  and  $W_+^s(0)$ .
- Invariant manifold,  $W_+^s(0)$ , of the origin in the  $\xi > 0$  dynamics (red) acts as target boundary condition for a shooting from  $W^{cu}(A^p)$ .
- Perturb from the generic free front, unfolding in  $c$  and  $\omega$  from  $c_{lin}$  and  $\omega_{lin}$ , obtaining a non-trivial intersection at  $(\xi, \omega) = (\xi_*, \omega_{tf}(c))$  (blue dot).



#### Geometric Desingularization

- Homogeneous variables:  $R = |A|^2, \quad S = |A_\xi|^2, \quad N = A\bar{A}_\xi$ ,

Blow-up in  $R$  and  $S$  directions:

$$z = \bar{N}/R = A_\xi/A, \quad R = |A|^2, \quad \text{and} \quad \tilde{z} = N/S = A/A_\xi, \quad S = |A_\xi|^2.$$

- Blow-up charts factor out  $S^1$ -invariance, phase portrait in  $\mathbb{R}_+ \times S^2$ :

$$z' = -(z+1)^2 + cz - i\omega + (1+i\gamma)R$$

$$R' = 2\text{Re}(zR) = R(z + \bar{z}).$$

- $S$  normally hyperbolic invariant manifold (with algebraic/exponential decay in the tangent/normal directions), use Fenichel's results on smooth foliations to perform heteroclinic matching.

### Pushed case

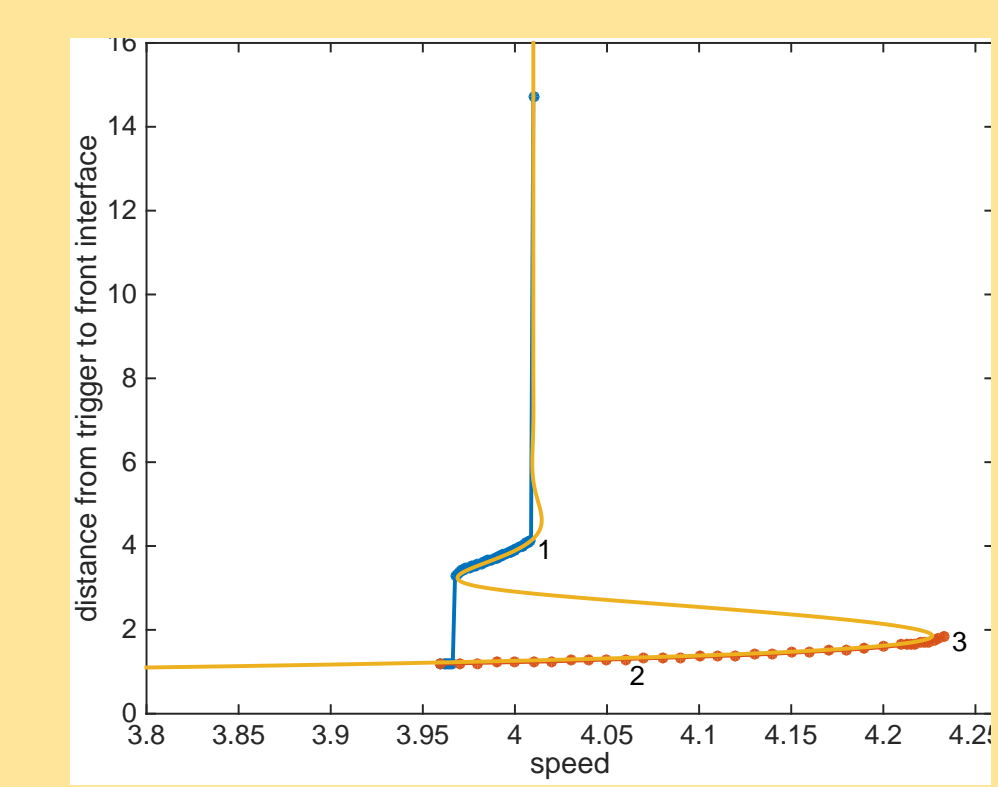
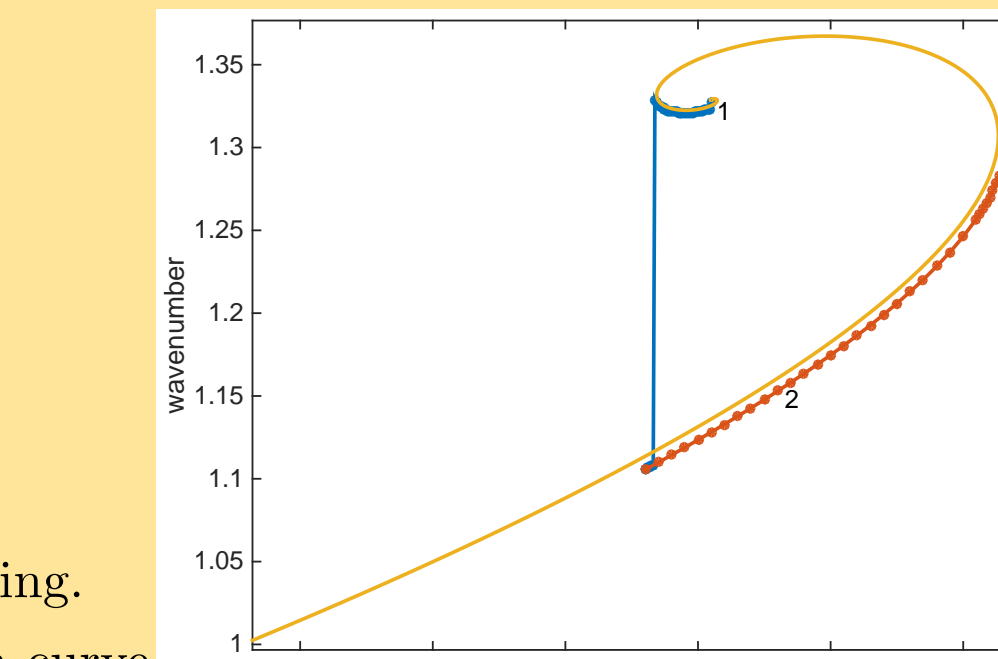
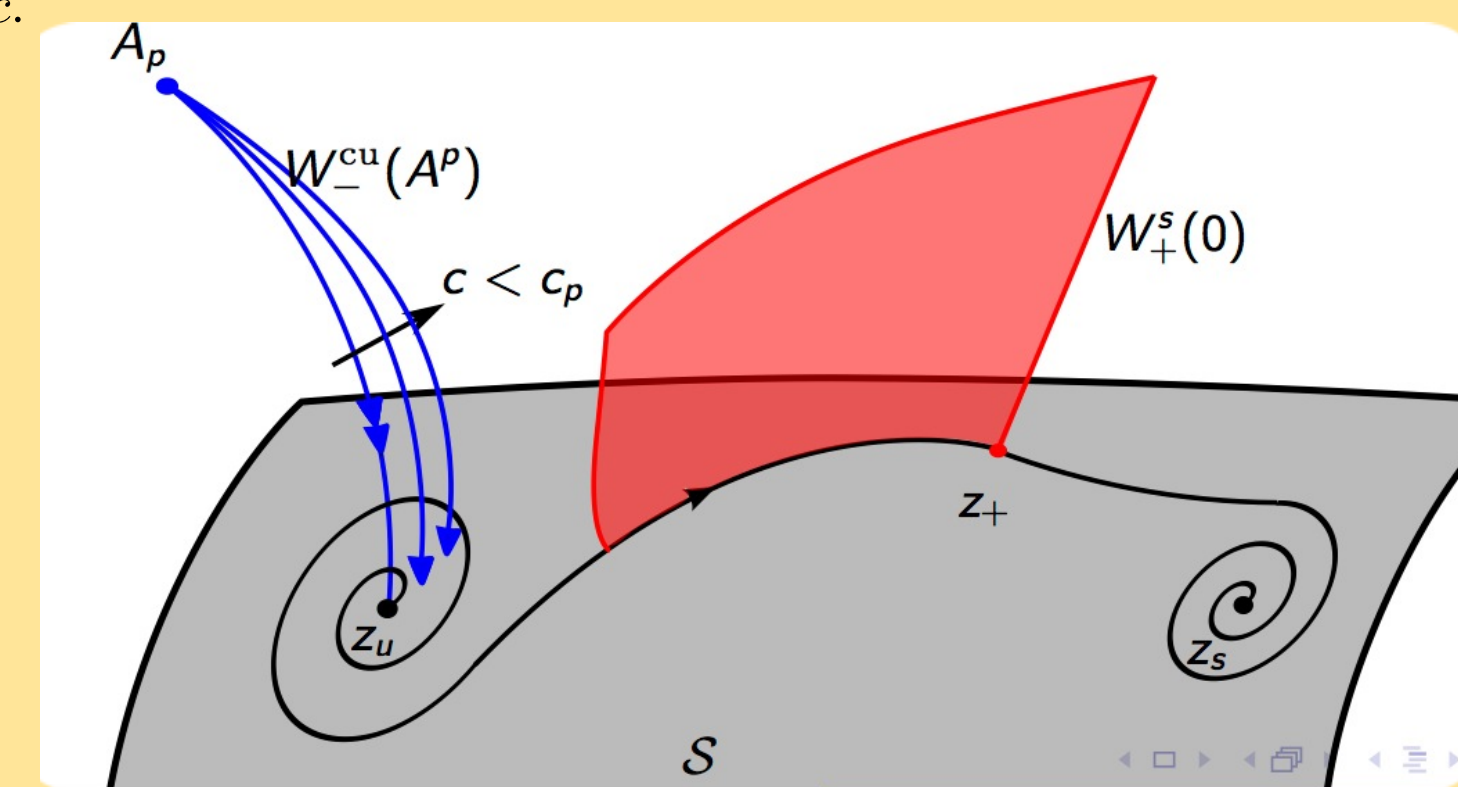
**Theorem 1.** Set  $\alpha, \beta, \gamma, \rho, \mu$  so free pushed front,  $A_{tf}$ , exists. For  $0 < c_{lin} - c \ll 1$ , there exists pushed trigger front  $e^{i\omega_{tf} t} A_{tf}(\xi)$  with  $\omega_{tf}(c)$  implicitly defined by

$$M\mu = \exp[(\nu_{ss} - \nu_{su})L + \mathcal{O}(\mu)], \quad L \gg 1, \quad \mu = (c_p - c, \omega_p - \omega)^T.$$

where  $M_{i,j} = \int_{-\infty}^{\infty} \langle \psi_i(\xi), \partial_j F(A_{tf}(\xi)) \rangle d\xi, \quad i = 1, 2, \quad j = c, \omega.$

#### Sketch of Proof:

- Perform same blow-up.
- Match oscillatory exponential tail with effective boundary condition
- Leads to hysteretic wavenumber selection, multi-stability, and front locking.
- Exponential expansions generically give logarithmic spiral as bifurcation curve in  $\omega, c$ .



### Cahn-Hilliard and beyond

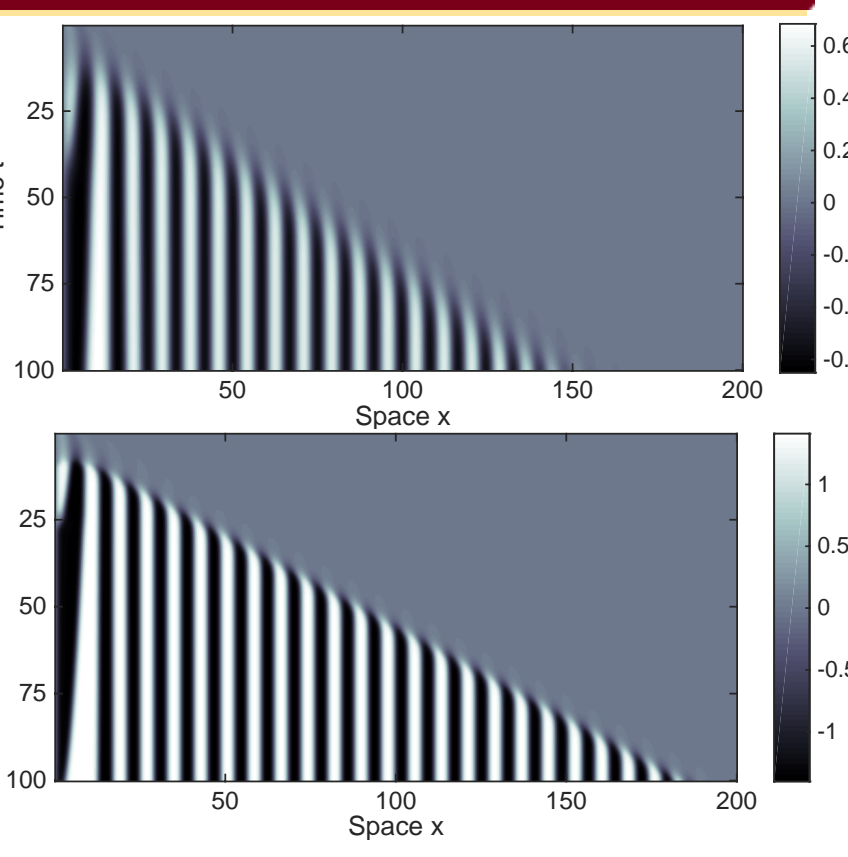
We demonstrate a general recipe for pushed trigger fronts, using the Cahn-Hilliard equation:

For  $\gamma > 0$ , free fronts  $u_{ff}$  are solutions of modulated traveling wave equation connecting  $u_0 \equiv 0$  at  $\xi = +\infty$  and  $u_p(kx - \tau)$  at  $\xi = -\infty$ ,

$$\omega_p u_\tau = -(u_{\xi\xi} + f(u))_\xi + c_p u_\xi, \quad x \in \mathbb{R}, \tau \in [0, 2\pi), \quad f(u) = u + \gamma u^3 - u^5$$

**Pulled** ( $\gamma < 0$ ) Parameters determined by linearized dynamics:  $c = c_{lin}, \omega = \omega_{lin}$ , and wavenumber determined by  $k_{lin} = \omega_{lin}/c_{lin}$ .

**Pushed** ( $\gamma > 0$ ) Parameters determined by nonlinear dynamics:  $c = c_p, \omega = \omega_p$ , and wavenumber determined by  $k_p = \omega_p/c_p$ .



### Triggered instabilities & spatial dynamics

Trigger  $\chi$  changes linear stability of  $u_0 \equiv 0$ , vary speed  $c$  near  $c_{lin}$  or  $c_p$ .

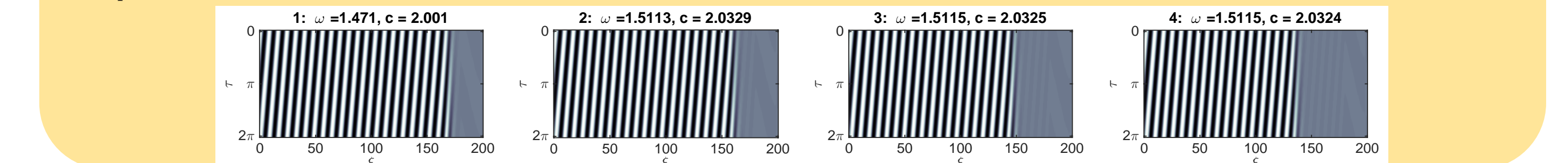
$$u_t = -(u_{xx} + f(x-ct, u))_{xx}, \quad f(\xi, u) = \chi(\xi)u + \gamma u^3 - u^5, \quad \chi(\xi) = \tanh(-\xi/\epsilon)$$

#### Spatial Dynamics formulation

Let  $U = (u, u_\xi, u_{\xi\xi} + \tilde{f}(\xi, u), (u_{\xi\xi} + \tilde{f}(\xi, u))_\xi)^T$  obtain

$$U_\xi = \mathcal{A}(\xi)U + G(U), \quad \mathcal{A}(\xi) = \begin{pmatrix} b_1(\xi) & I_3 \\ -\omega \partial_\xi & b_2(\xi) \end{pmatrix}, \quad G(U) = (0, -\gamma u^3 + u^5, 0, 0)^T$$

- Evolution on  $X = H^3(\mathbb{T}) \times H^2(\mathbb{T}) \times H^1(\mathbb{T}) \times L^2(\mathbb{T})$ ,  $D(\mathcal{A}) = H^4(\mathbb{T}) \times H^3(\mathbb{T}) \times H^2(\mathbb{T}) \times H^1(\mathbb{T})$
- Make system autonomous, by requiring  $\chi$  solve  $c\chi' = 1 - \chi^2$ .
- $u_{ff}$  is a heteroclinic orbit in  $\chi = 1$ -subspace,
- $u_{pr} \equiv 0$  is heteroclinic connecting  $(U, \chi) = (0, 1)$ , to  $(0, -1)$ .



### Hypotheses & result: pushed case

#### Assume:

- (i).  $\mathcal{A}$  has spectral gap demarcated by  $\nu_{ss}$  and  $\nu_{su}$ , both with neg. real part
- (ii). Existence of pushed free front  $q_1$  in  $\chi \equiv 1$  system, with decay  $\sim e^{\nu_{ss}\xi}, \xi \rightarrow \infty$
- (iii). Invertibility of Melnikov matrix assoc. with  $\partial_\xi q_1, \partial_\omega q_1$
- (iv). Existence of front  $q_2$  connecting  $(U_*, \chi_+) := (0, 1)$  to  $(0, -1)$
- (v).  $q_1$  and  $q_2$  form co-dimension-2 heteroclinic chain
- (vi). Inclination property:  $T_{q_2(0)} W^{ss}(U_*) \not\subset T_{q_2(0)} W^s(\tilde{U}_*)$
- (vii). Well-posedness of variational system

**Theorem 1.** For  $\mu = (c - c_p, \omega - \omega_p)$ , and  $L \gg 1$ , there exists 1-parameter family of pushed trigger fronts  $u_{tf}(\xi, t; \mu; L)$  where the parameter curve  $\mu_* : [L, \infty) \rightarrow \mathbb{R}^2$  approaches the origin with leading order asymptotics  $e^{(\nu_{ss} - \nu_{su})L}$ .

⇒ snaking, multi-stability of fronts, front locking, hysteresis

### Method of proof

Study variational equations about  $q_1$ , and  $q_2$ :

$$\frac{d}{d\xi} w_i = A_i(\xi)w_i + g_i(\xi, w_i; \mu), \quad \xi \in \mathbb{R}, \quad A_i(\xi) := D_U F(q_i(\xi)), \quad g_i(\xi, w_i; \mu) := F(q_i(\xi) + w_i; \mu) - F(q_i(\xi); 0) - A_i(\xi)w_i.$$

where  $F(U) = \mathcal{A}(\xi)U + G(U)$ .

#### Gluing-matching (à la Rademacher and Lin)

- (Silnikov Sol.) Use var. of const. to prove existence of solutions  $w_i$  near  $q_1, q_2$  lying in exponentially weighted function spaces with boundary conditions.

$$P_1^{ss}(0)w_1(0) = s_1, \quad P_1^{su}(L)w_1(L) = u_1, \quad P_2^{ss}(-L)w_2(-L) = s_2, \quad P_2^{su}(0)w_2(0) = u_2, \quad (0.1)$$

- (Gluing): Use gluing condition  $w_2(-L) - w_1(L) = q_1(L) - q_2(-L)$ , to solve for the "outer" boundary variables  $\mathfrak{W}^0 := (s_1, u_2)$  in terms of the "inner" boundary variables  $\mathfrak{W}^L := (s_2, u_1), L$ , and  $\mu$ .
- (Transverse intersection): Match the solution  $w_2(\mathfrak{W}^0; \mu, L)$  with  $W^s(\tilde{U}_*(\mu))$  at some transverse section  $\Sigma_2$  of  $q_2(0)$ .
- (Non-Transverse intersection): Match the solution  $w_1(\mathfrak{W}^L; \mu, L)$  with  $W^{cu}(U_p)$  at a transverse section  $\Sigma_1$  of  $q_1(0)$  by first solving the matching condition in  $E_1 := T_{q_1(0)} W^{ss}(0) + T_{q_1(0)} W^{cu}(U_p)$ . Then solve the condition in the complement  $E_1^\perp$  using Melnikov integrals.

