TUTORIAL M-IV:

Computing Lyapunov Exponents to characterize chaos and bifurcations of tori in MatContM

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MatContM is available at `http://sourceforge.net/projects/matcont/files/matcontm/`. This tutorial is tested on Matlab R2017b with matcontm5p4. In this tutorial we compute Lyapunov exponents for three different examples.

We start with the planar Hénon map. It is one of the most studied maps with a chaotic attractor. Through the geometric construction of Smale's horseshoe of stretching and folding back, this example exhibits chaotic dynamics. Next we consider the well-known logistic map as it has the classical route to chaos via successive period-doubling bifurcations. Finally, we look at an economic model for asset prices, as we can monitor some bifurcations of invariant curves. While Lyapunov exponents are useful indicators, we should be careful in their interpretation. While a positive largest exponent suggests chaos, it does not substitute a proof of chaotic dynamics. Similary a zero Lyapunov exponent suggests an invariant curve but the numerical computation may not converge to zero quickly. So this has to be interpreted with care.

1 Example 1; Hénon Map

For this example we use the Generalized Henon Map created before, TUTORIAL M-III, using $R = S = 0$. We will compute the Lyapunov exponent for a single set of parameter values. We will explain how various options in the algorithm affect speed and accuracy.

1.1 Preparation and Input

Load the map via **System|Systembrowser**. Set the Point type **Main|Type|Point** and select the computation of **Starter|Initializer|Lyapunov Exponents QR-method**. In the Starter window we set $x = 0.2, y = 0.3$ and $a = 1.4$ and $b = -.3$. Also in the Starter window, we choose settings Lyapunov steps=100000, normsteps=10, report every x normalizations=2000. Now press **Compute|Forward**. The Output window appears where every 20000 (=10*2000) steps the intermediate result is shown. After 2.8 seconds the computation was succesfully completed.

1.2 Results

In the workspace we now find a new variable "lyapunovExponents", an array with two numbers. These are the computed Lyapunox exponents and we have $L_1 = .4205$ and $L_2 = -1.6245$. If we increase the number of steps, our estimate will eventually converge, independent of the initial point. The first iterates may still just be transients though before they reach the attractor. Therefore, it may be advantageous to start the computation only after some number of transient steps such that only the expansion/contraction near the attractor is sampled.

In Figure 1(right) we have plotted the ongoing estimates for the Lyapunov exponents. One may observe the fluctyations which are still visible towards high step numbers. It is clear that a more accurate result requires many steps. As a sort of error we take the difference of the minimum and maximum of the ongoing estimates during the second half of the computation. Then we get $L_1 = 0.420 \pm .002$ and $L_2 = -1.625 \pm .002$. There is a clear trade-off between speed and accuracy. Setting the number of Lyapunov steps=1e7 ($= 10^7$), we find $L_1 = 0.4190 \pm .0002$ and $L_2 = -1.6230 \pm .0002$.

The algorithm determines the expansion/contraction in every direction. If the Lyapunov vectors become completely aligned, then the algorithm the most contractive direction become less reliable. Therefore the vectors need to be orthonormalized. This is computationally expensive and so this is done not at every step. To avoid numerical overflow, however, the number of steps to orthonormalization cannot be set too high. In this example, normsteps=20 would be too high as $\exp(L_1 - L_2)^{20} \approx 6e + 17$. After 20 steps, the second Lyapunov vector will be numerically the same as the first and any numerical result would be erratic.

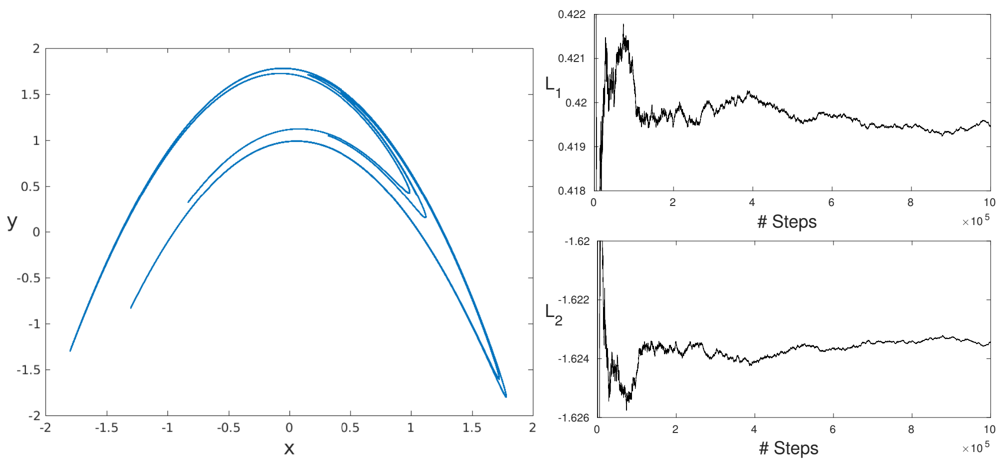

Figure 1: Left: Phase plot of the Hénon attractor. Right: The ongoing estimates of the two Lyapunov exponents shown separately as the orders of magnitude differ.

2 Example 2; Logistic Map

For the logistic map we want to show how the exponents can be computed for a range of parameter values. That is, as the attractor changes we can monitor how the exponents change.

2.1 Preparation and input

The logistic map is given by

$$F_1 : x \mapsto ax(1-x).$$

We create a new system with model name logistic with coordinate x and parameter a, see Figure 2 (left). We set $x = 0.1$ as initial condition as otherwise we stay in the origin forever. For the

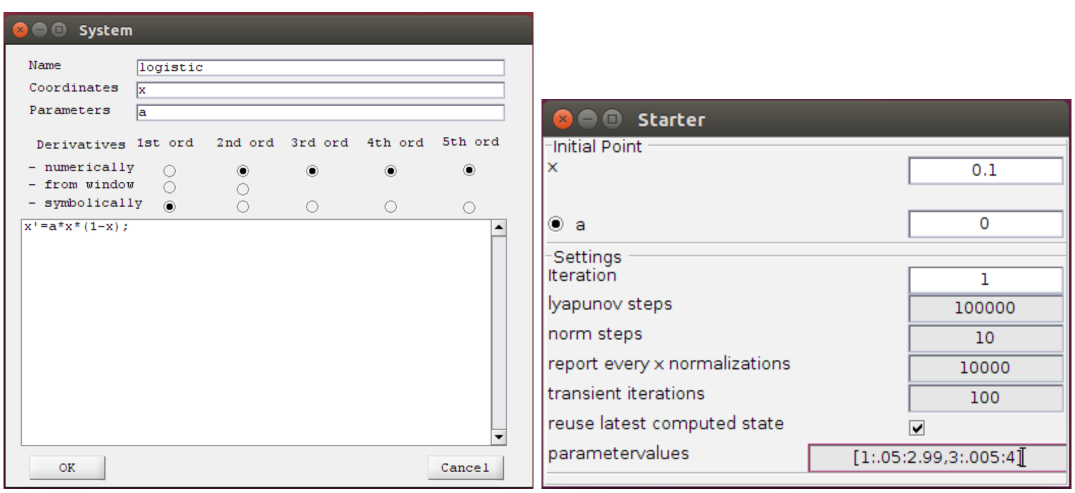

Figure 2: Left: The system input for the logistic map. Right: The settings for the computation with active parameter a (button ticked). The array for the parameter values is automatically expanded.

parameters we choose a concatenated array of values to monitor how the single exponent changes as the parameter a varies. We take non-equidistant steps to speed up the computation. We set $a = [1 : .05 : 2.99, 3 : .005 : 4]$, see Figure 2 (left), and set a as an active parameter by clicking the button. Next press **Compute|Forward**. This takes a while, about 1-2 minutes, and then the results are written to the workspace into a structure lyapunovExponents with two fields, the parameter and the exponents. Finally, we plot our results as follows.

```
figure;
a=lyapunovExponents.a;
plot(a,lyapunovExponents.exponents,a,0*a);
xlabel('a');ylabel('\lambda')
```

2.2 Results

If all is well, you now have Figure 3(left). We have also plotted another familiar graph for this map, which is the coordinate x found by simply iterating the map, see Figure 3(right). Until $a = 1$, the origin $x = 0$ is the only fixed point and all orbits converge to it. At $a = 1$ we encounter a branching point where a positive fixed point $x^* = (a-1)/a$ appears. At $a = 2$, the multiplier of the fixed point x^* equals 0. This is visible as the Lyapunov exponent has a vertical asymptote at $a = 2$. This phenomenon is referred to as superstability in work on "shrimps" (Vitolo, Glendinning, Gallas, 2011). At $a = 3$, the first period-doubling occurs which is visible as the Lyapunov exponent increases to zero, and next decreases again. The increase may be interpreted that in that corresponding direction the attractor becomes less attracting until it loses stability. The next period-doubling occurs at $a = 1 + \sqrt{6} \approx 3.45$ where the exponent increases to zero and decreases again. This repeats until $a \approx 3.57$ where chaos sets in. Here we have positive Lyapunov exponents alternating with regions where we have stable cycles.

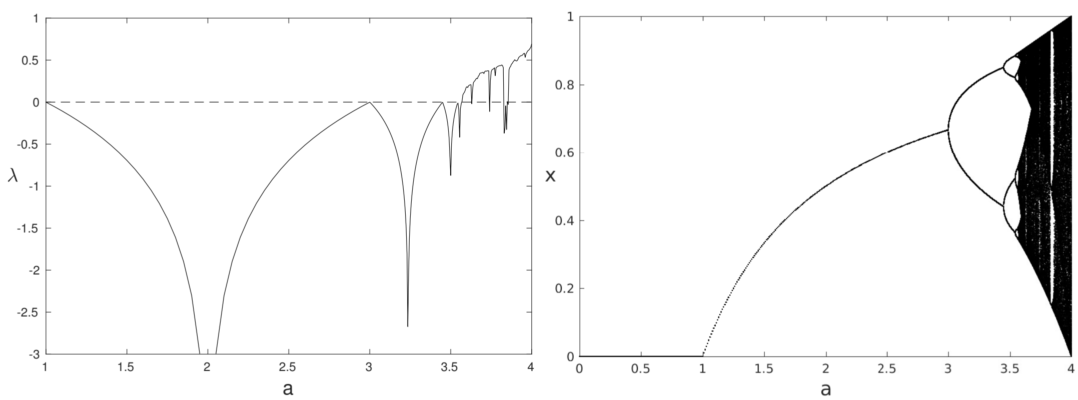

Figure 3: Left: the Lyapunov exponent as a function of the parameter a. Right: The set of x-values visited by the attractor.

3 Example 3; Resonances and Quasi-Periodic bifurcations

In this example we consider the dynamics of a model of volatility in asset prices, see Gaunersdorfer, Hommes and Wagener 2008, with more details in the working paper . In memoryless form and with reduced parameter space the model may be written as

$$F := (x_1, x_2, x_3, x_4) \mapsto (1 - n(x))vx_1 + n(x)(x_1 + g(x_1 - x_2)), x_1, x_2, x_3) \,,$$

where

$$n(x) = \frac{e^{-x_1^2} e^{-bu_2}}{e^{-bu_1} + e^{-bu_2}},$$

with

$$u_1 = (x_1 - vx_3)^2 \text{ and } u_1 = (x_1 - x_3 - g(x_3 - x_4))^2.$$

The origin is a fixed point of the model and has a Neimark-Sacker bifurcation for $g = 2R$. For $v \approx 0.45$ the Neimark-Sacker bifurcation is degenerate. Here a quasi-periodic saddle-node bifurcation leading to a stable large "outer" invariant curve. We focus on the computation and explaining the computational results and do not interpret the dynamics for the application. The idea here is to follow the invariant curve as we change a parameter. We encounter resonances and a quasi-periodic saddle-node bifurcation, where the invariant curve terminates to exist. We want to show how the Lyapunov exponents may be classified to demarcate bifurcations that are otherwise hard to find.

3.1 Preparation and Input

We specify the system 'volatility' with coordinates $x1, x2, x3, x4$ and parameters g, v, R, b as in Figure 4(left). Next we select **Main|Type|Initial Point|Point** and **Main|Type|Curve|Compute Lyapunov Exponents (QR....)**. In the Starter Window We set the initial condition $x1 = 3, x2 = 3, x3 = 1, x4 = 1$ and parameters $g = 2.0, v = .8, R = 1.01, b = 8$. With these settings we start in the bistable region on the large invariant curve. Next we set g as Active Parameter, normsteps=5, lyapunov steps=1e6 and $g = 2 : -.0005 : 1.7$, see Figure 4(right). Next press **Compute|Forward**. This may take a while.

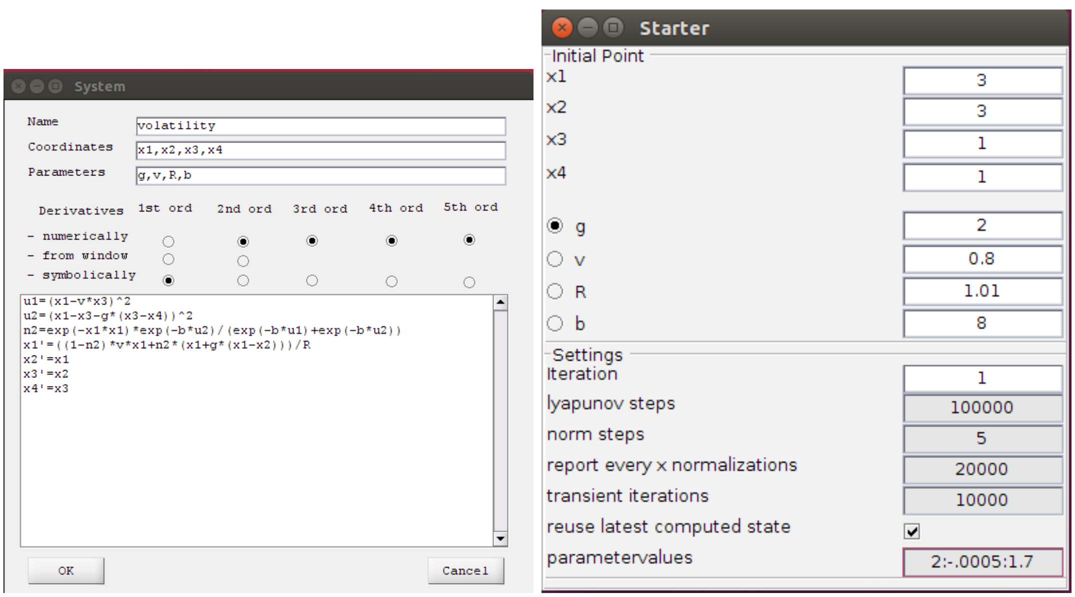

Figure 4: Left: The system input for the map. Right: The settings for the computation.

3.2 Results

Using the same plot commands as for the logistic map, but now with g instead of a, we obtain Figure 5(bottom). As $\lambda_{3,4}$ have a different order of magnitude we also show a plot with the range restricted to $\lambda_{1,2}$. As we decrease from $g = 2.0$ to $g \approx g_c := 1.72633$, we see that $\lambda_1 \approx 0$ most often. This indicates an invariant curve. We could use a threshold $|\lambda_1| < 10^{-4}$ for classification in this case, while all other exponents are more negative here. When a resonance is encountered, the dynamics on the invariant curve reduces to a cycle with a high period. In the figure we see resonances of period 17, 18, 19 and 20. The stability of the cycles inside the resonance tongues can be observed by all four exponents being negative. As we decrease g further, there is a critical value g_c where we see that λ_2 comes closer and closer to zero. Here the invariant curve exhibits a quasi-periodic saddle-node bifurcation. Decreasing g further, we observe a sudden drop in the exponents. As the invariant curve is lost as attractor, the orbit now collapses to the origin. The origin has two zero multipliers and a complex pair within the unit circle, and hence only two Lyapunov exponents are well-defined as the others are $-\infty$ (and hence not drawn). The sudden change can be monitored to find a more precise bifurcation value. Summarizing, this example shows how Lyapunov exponents may hint at quasi-periodic bifurcations. A stronger statement requires a more rigorous treatment involving the computation of the normal behaviour of the invariant curve.

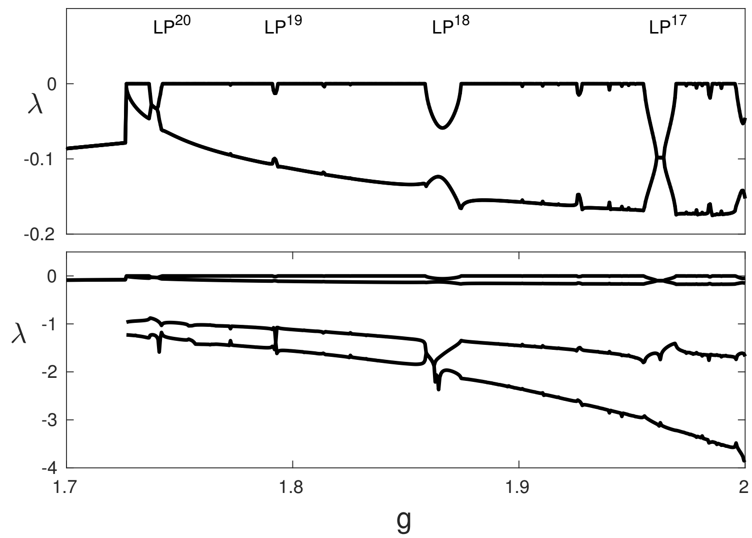

Figure 5: Evolution of the Lyapunov exponents as g is decreased from 2.0 to 1.7. Bottom figure shows all 4 exponents, while top zooms in on the two exponents close to zero.