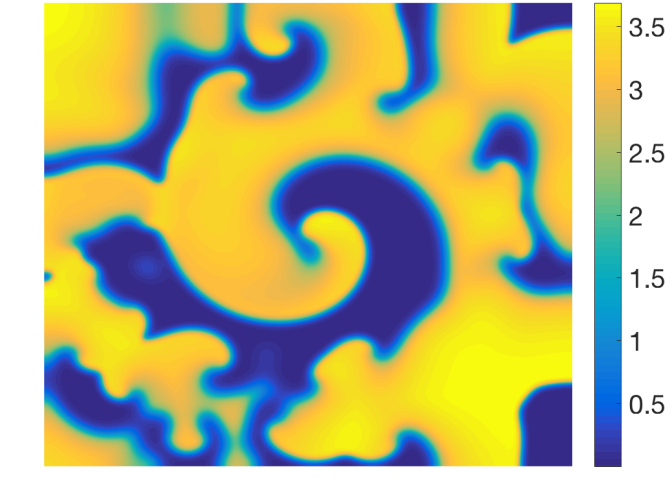
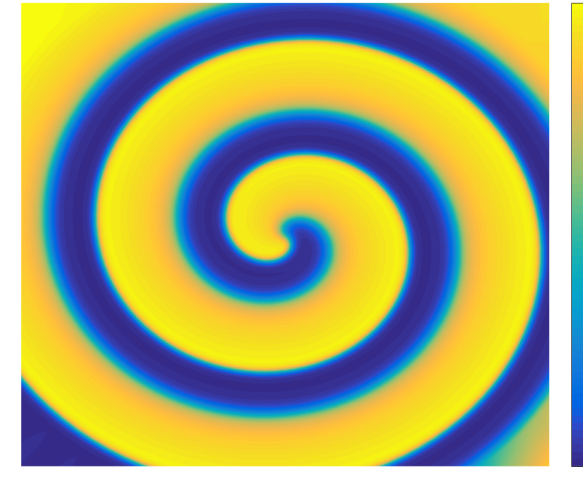


Spiral Waves in Cardiac Arrhythmias

- Cardiac Arrhythmia refers to abnormal electrical activity in the heart.



Ventricular Tachycardia: (VT)
Reentrant spiral waves create self-sustained oscillations.

Ventricular Fibrillation: (VF) Spiral wave breakup leads to unorganized self-sustained electrical activity.

- VF may lead to sudden cardiac death, which is responsible for > 350,000 deaths/year.
- Alternans** is a marker for sudden cardiac death.

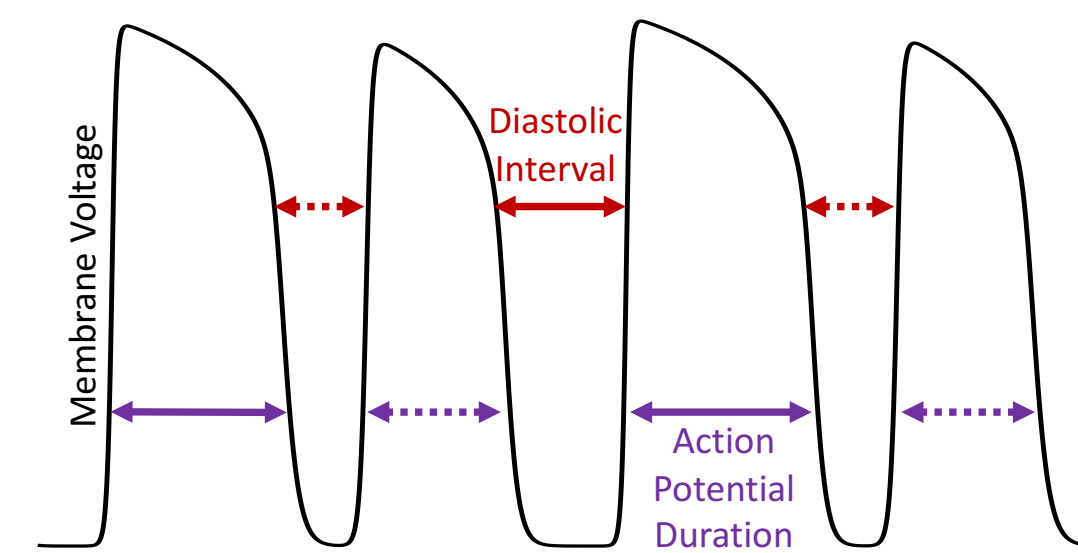


Figure: Cartoon of alternans. Membrane voltage exhibits periodic variation in length of action potential duration and diastolic interval.

Goals

The aim of this research is to investigate what spectral properties can tell us about the stability of spiral waves in cardiac arrhythmias, in particular alternans instability.

- Understand and illustrate properties of spiral spectra.
- Relate spectral properties to alternans instability observed in spirals.

The Karma Model

A simple reaction-diffusion cardiac model that exhibits alternans.

$$\begin{cases} E_t = \gamma \Delta E + \frac{1}{\tau_E} \left(-E + [E^* - n^M][1 - \tanh(E - E_h)] \frac{E^2}{2} \right) \\ n_t = \delta \Delta n + \frac{1}{\tau_n} \left(\frac{1}{1 - e^{-Re}} \theta(E - E_n) - n \right) \end{cases}$$

- $E(x, t)$ describes membrane voltage and $n(x, t)$ provides slower dynamics.
- $E_n, E_h, E^*, \delta, \tau_E, \tau_n \in \mathbb{R}$ control excitable threshold and fast/slow timescale.
- $Re \in \mathbb{R}$ controls slope of restitution curve.

Written as a system in polar coordinates, the model is

$$U_t = D \Delta_{r,\phi} U + F(U), \quad U = \begin{pmatrix} E \\ n \end{pmatrix} (r, \phi), \quad D = \begin{pmatrix} \gamma & 0 \\ 0 & \delta \end{pmatrix}.$$

Rigidly rotating spiral waves, $U^*(r, \psi)$, are stationary solutions in a rotating polar frame, $(r, \phi) \rightarrow (r, \psi) = (r, \phi - \omega t)$

$$0 = D \Delta_{r,\psi} U^* + \omega U_\psi^* + F(U^*).$$

Spirals tend to 1D periodic **asymptotic wave trains**, U^∞ , as $r \rightarrow \infty$

$$U^*(r, \psi) \rightarrow U^\infty(\kappa r + \psi) = U^\infty(\xi), \quad U^\infty(\xi) = U^\infty(\xi + 2\pi).$$

Wave trains are stationary solutions of

$$U_t = \kappa^2 D U_{\xi\xi} + \omega U_\xi + F(U).$$

Acknowledgements

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Types of Spectra

Temporal Eigenvalues, λ , describe temporal growth of perturbations

$$\mathcal{L}U = D \Delta_{r,\psi} U + \omega U_\psi + F'(U^*)U = \lambda U.$$

Spatial Eigenvalues, ν , describe the spatial growth of eigenfunctions.

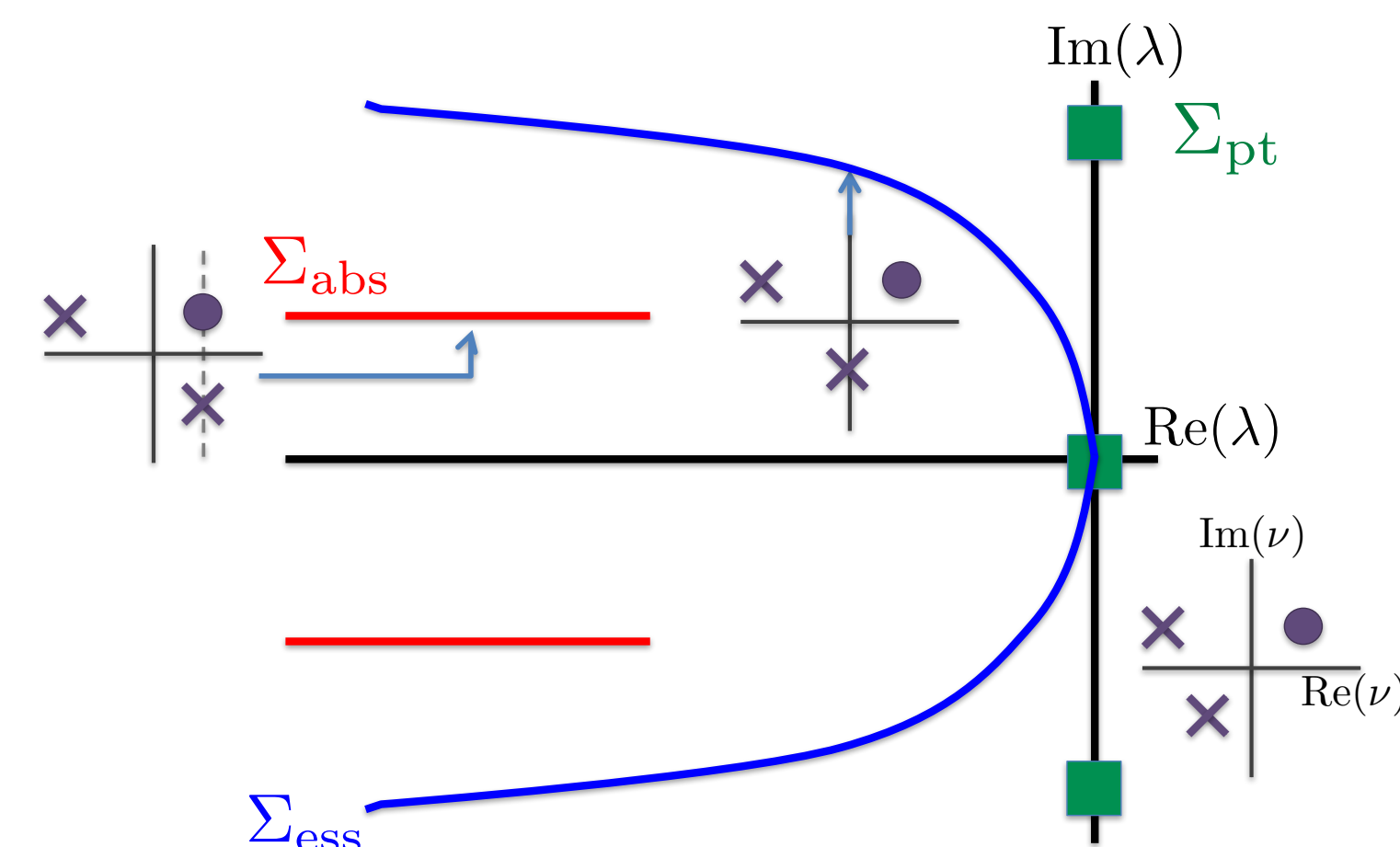


Figure: Cartoon of essential, absolute, and point spectra. Inserts show distribution of spatial eigenvalues.

On the plane:

- Point spectrum, Σ_{pt}
- Essential spectrum, Σ_{ess}
- $(\lambda - \mathcal{L})$ is not Fredholm
- $\nu \in i\mathbb{R}$

On bounded domain:

- Point spectrum, Σ_{pt}
- Absolute spectrum, Σ_{abs}
- Limit of discrete spectrum as domain $\rightarrow \infty$
- No longer separate stable/unstable spatial eigenvalues

Dispersion relations of spiral, $\lambda_*(\nu_*)$, and wave train, $\lambda_\infty(\nu_\infty)$, are related via

$$\lambda_*(\nu_*) = \lambda_\infty(\nu_\infty) - \omega \nu_\infty + i \omega \ell, \quad \ell \in \mathbb{Z}, \quad \nu_* = \kappa \nu_\infty.$$

Alternans is Preceded by Meandering

- Known that Hopf bifurcation leads to meandering.

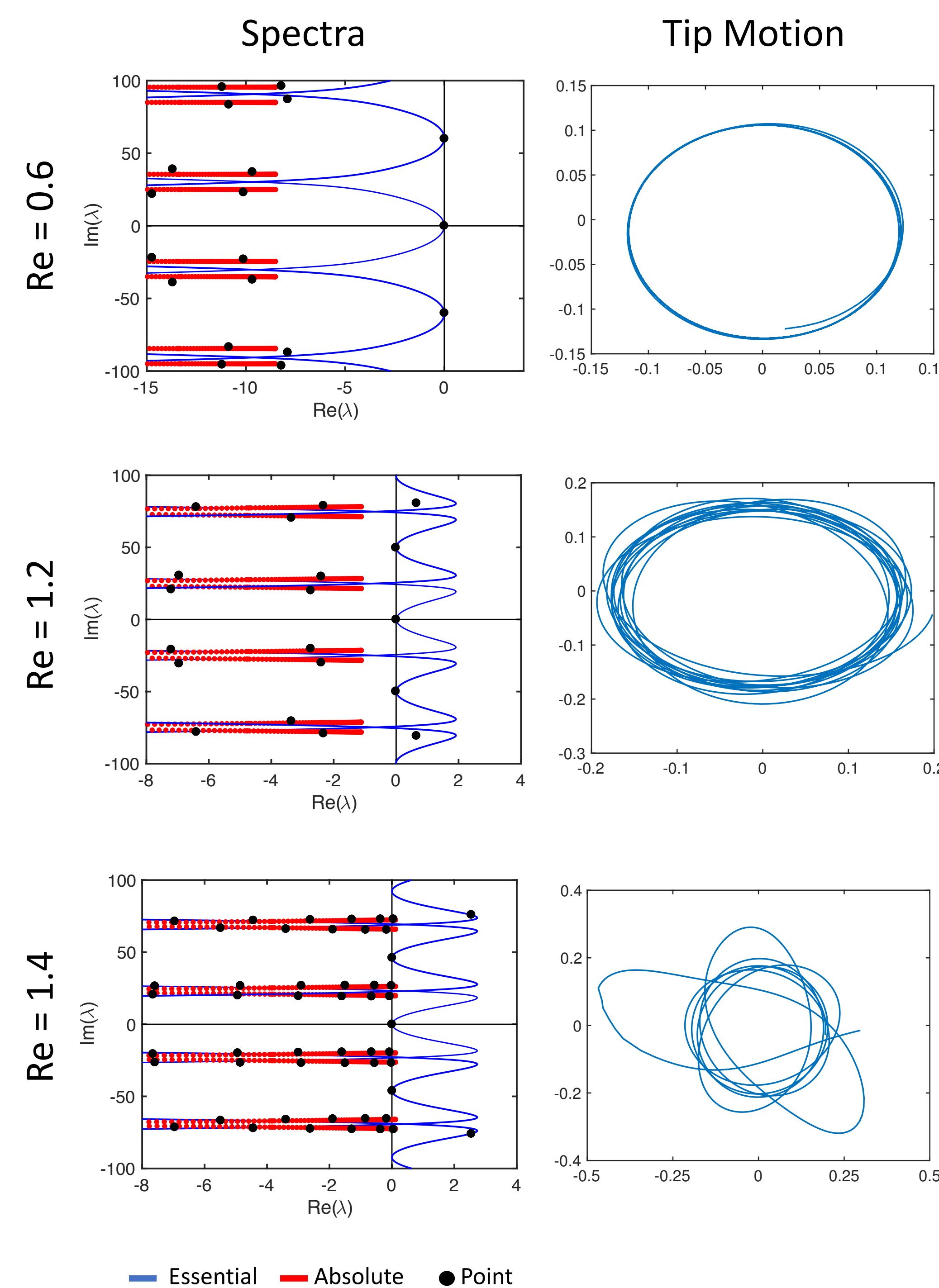


Figure: Spectra and tip motion of spirals in the Karma Model. Point spectrum calculated from spiral on 5 cm bounded disk, absolute and essential spectrum from wave trains. Additional parameters: $E_h = 3, E_n = 1, E^* = 1.5414, \tau_E = 0.0025, \tau_n = 0.25, \gamma = 1.1, \delta = 0.1$.

Unstable Eigenfunctions

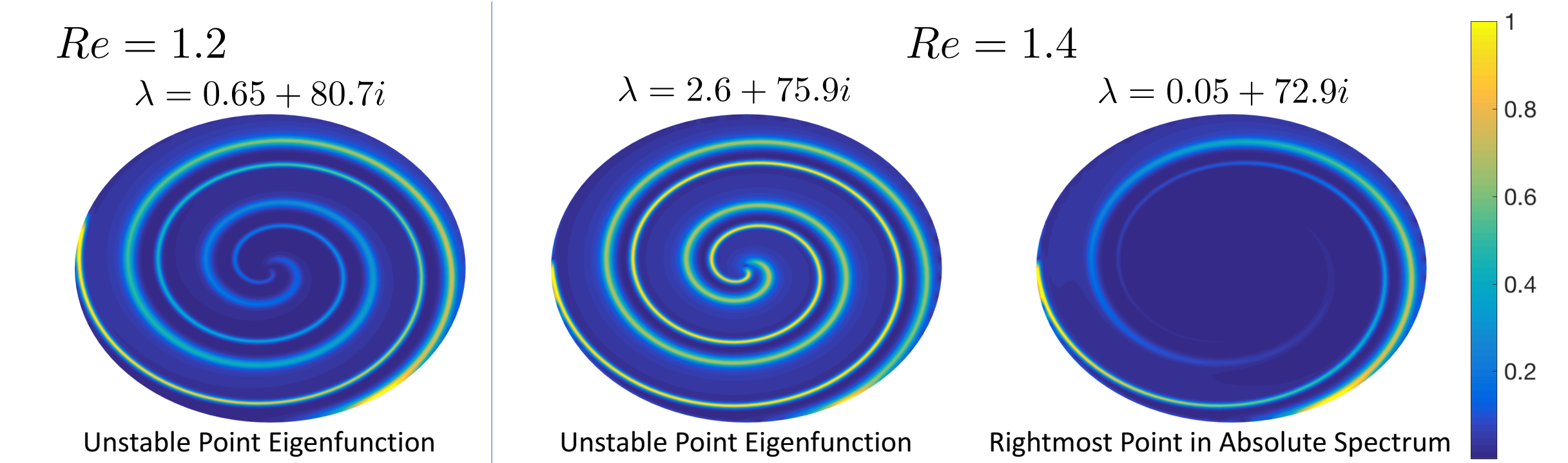


Figure: Absolute value of E-component of unstable eigenfunctions for radius 5 cm spiral.

- Growth toward boundary in $Re = 1.2$ unstable eigenfunction.
- Unstable point eigenfunction in $Re = 1.4$ interacts with essential spectrum and has global behavior.
- Eigenfunctions in absolute spectrum are localized away from spiral core.

Point and Absolute Spectrum Leads to Planar Alternans

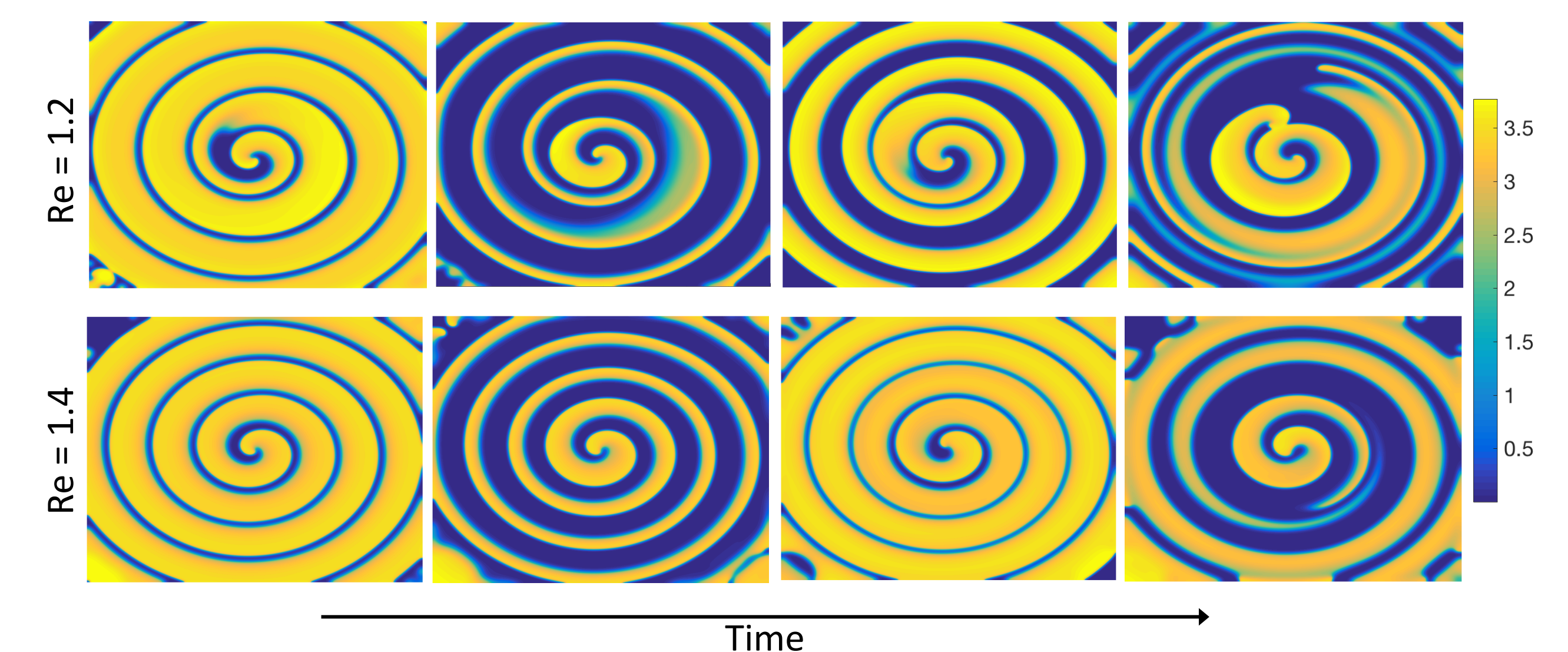


Figure: Spiral breakup due to alternans in the Karma Model on a 16 cm x 16 cm square with Neumann boundary conditions. Top row shows $Re = 1.2$, bottom is $Re = 1.4$. Color-bar indicates the membrane voltage. Solutions evolved on fourth-order finite-difference spatial grid using Crank-Nicholson and Adams-Bashforth IMEX scheme.

Conclusions

- Spiral breakup occurs as bands collide and form conduction blocks.
- Form of unstable eigenfunctions shows expansion/compression of spiral bands associated with alternans.
- Alternans instability likely caused by unstable eigenfunctions in the point spectrum.

Future Work

- Use 1D eigenfunctions to learn about and predict the shape of instabilities.
- Evaluate contributions to spiral break up from point and absolute spectrum.
- Determine if Hopf bifurcations are super or subcritical.
- Analyze case when one or more variables are diffusionless.
- Investigate alternans instability in other cardiac models.

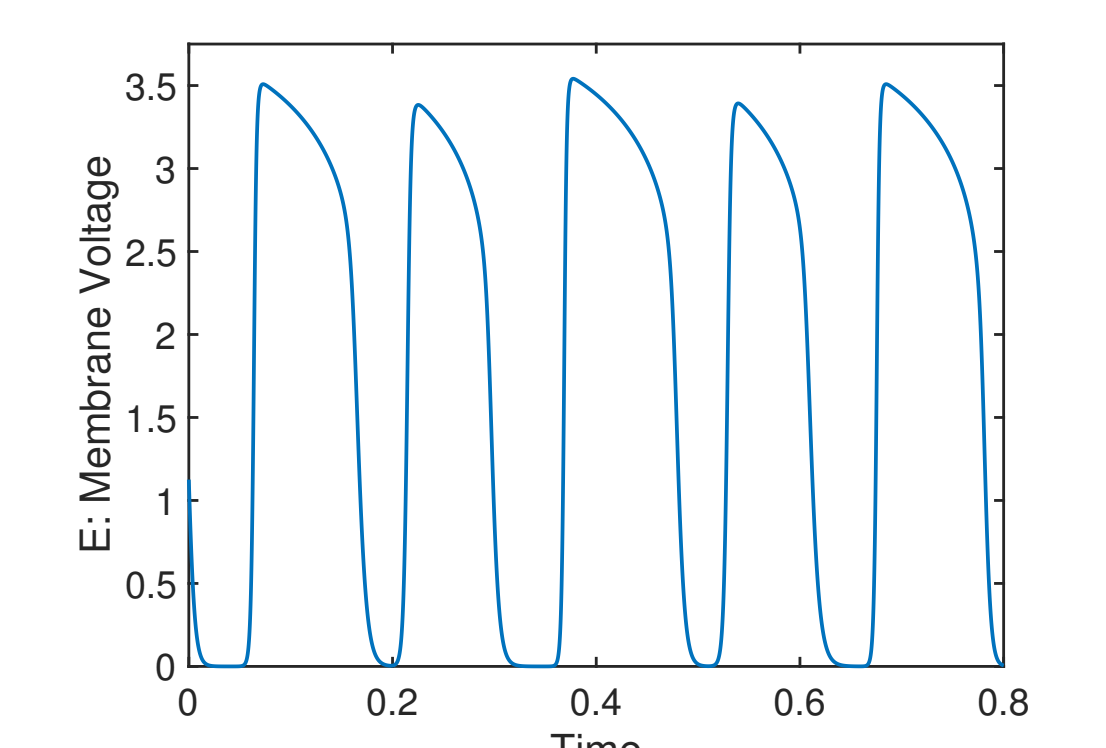


Figure: Alternans in 1D Karma Model.

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